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EXACT AND ASYMPTOTIC SOLUTIONS OF THE CAUCHY PROBLEM

by

Donald Ludwig

September 9, 1959

Institute of Mathematical Sciences

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ABSTRACT

This paper is concerned with the Cauchy problem for linear hyperbolic equations or systems with variable coefficients. The main problems considered are the propagation of singularities, focussing of singularities, and the construction of fundamental solutions. The methods which are employed correspond closely to the methods of geometrical optics and related fields.

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EXACT AND ASYMPTOTIC SOLUTIONS OF THE CAUCHY PROBLEM

1. Introduction

A. Statement of the problem.

This paper is concerned with the Cauchy problem for linear hyperbolic systems of k partial differential equations, each of order m, in k functions (which we write as the vector $u(x)$). All the coefficients of the equations are generally assumed to be smooth (C^∞) functions of $n+1$ independent variables (which we denote by the vector x). The characteristic directions of the system need not be distinct, but their multiplicity must remain constant within the region under consideration. Our main objective is the study of the properties of the solution in cases in which the Cauchy data is singular in the sense of distributions.

We shall give detailed consideration to two problems of this type which are of particular interest for mathematical and physical applications: that in which the data is singular along an $(n-1)$-dimensional manifold in the initial surface, and that of the fundamental solution (in which case the data is singular at a single point on the initial surface). Although the existence and uniqueness of solutions with distributions as Cauchy data has already been proved (see J. Leray [8] and R. Courant and P. D. Lax [4]), we shall incidentally obtain a new proof in these two cases in the course of our construction of the singular part of the solution.

This work was inspired by the paper by R. Courant and P. D. Lax [4] which deals with solutions with jump discontinuities in their Cauchy data, and the fundamental solution.1/ This paper extends the methods and the results of Courant and Lax to cover a much larger class of problems. Their methods are clarified and simplified by a systematic use of the concepts and results of the theory of distributions.

In the preparation of this paper, the author has benefitted from the instruction, assistance and advice of Richard Courant, Joseph B. Keller and Peter D. Lax. Peter Ungar made many suggestions to improve the manuscript.

B. Method and results.

The method of this paper may be described as an application of the methods of geometrical optics to hyperbolic initial value problems. The transition from wave optics (represented by the partial differential equation) to geometrical optics (which involves only ordinary

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1/ Methods of Mathematical Physics, II, by R. Courant and D. Hilbert, to appear shortly, will contain a comprehensive treatment of the problems and methods of [4], and a detailed treatment of certain topics discussed in this paper. The propagation of jump discontinuities is also treated in R. Lewis [9].

differential equations) is effected by means of an asymptotic expansion. For example, we may look for formal solutions of the wave equation of the form

$$v(t,x) = e^{i\omega\phi(t,x)} \sum_{j=0}^{\infty} \frac{a^j(t,x)}{(i\omega)^j} . \tag{1.1}$$

Here we neither require nor expect that the series shall converge, but only that v shall be an approximate solution for sufficiently large values of ω. In other words, the partial sum

$$v^J(t,x) = e^{i\omega\phi(t,x)} \sum_{j=0}^{J} \frac{a^j(t,x)}{(i\omega)^j} , \tag{1.2}$$

shall differ from some solution of the wave equation by a term of order less than ω^{-J}.

These conditions, for $J = 0,1,\ldots,$ immediately imply relations which $\phi(t,x)$ and $a^j(t,x)$ $(j = 0,1,\ldots)$ must satisfy. In particular, $\phi(t,x)$ must be a solution of the characteristic equation

$$\phi_t^2 = \sum_{\nu=1}^{n} \phi_{x^\nu}^2 , \tag{1.3}$$

The characteristic equation can be solved by the method of characteristics: certain curves (called bicharacteristics or rays) are introduced, and ϕ is determined by means of ordinary differential equations along the rays. Similarly, the coefficients $a^j(t,x)$ may be determined by means of ordinary differential equations (called transport equations) along the rays.

Again, it should be emphasized that these asymptotic solutions are generally not exact solutions of the partial differential equation. However, asymptotic solutions are usually easier to compute and manipulate than the corresponding exact solutions. Frequently one or two terms of the asymptotic expansion provide a useful and convenient approximation.

The scope of the methods of geometrical optics can be extended in two ways. First, such asymptotic expansions can be constructed for any partial differential equation or system with real characteristics, which satisfy certain algebraic conditions. For a wide class of hyperbolic equations or systems, the Cauchy problem has an asymptotic solution provided that the Cauchy data has an asymptotic expansion.

Secondly, we observe that, in deriving the equations which determine $\phi(t,x)$ and $a^j(t,x)$, only a single property of the exponential function was used: namely

$$(1.4) \qquad \frac{d}{d\phi}\frac{e^{i\omega\phi}}{(i\omega)^j} = \frac{e^{i\omega\phi}}{(i\omega)^{j-1}} \quad .$$

It follows that if we have a set of functions $\left\{f_j(\phi)\right\}$ which satisfy the relations

$$\frac{d}{d\phi} f_j(\phi) = f_{j-1}(\phi) , \tag{1.5}$$

then the expression

$$u(t,x) = \sum_{j=0}^{\infty} f_j(\phi(t,x)) a^j(t,x) , \tag{1.6}$$

will also formally satisfy the differential equation. Now such a set of functions is essentially determined by the function $f_o(\phi)$; we have

$$\left\{ \begin{aligned} f_1(\phi) &= \int f_o(\phi)\, d\phi \\ f_2(\phi) &= \int f_1(\phi)\, d\phi , \\ &\vdots \end{aligned} \right. \tag{1.7}$$

Moreover, the function $f_o(\phi)$ may be chosen arbitrarily. Thus, once $\phi(t,x)$ and $a^j(t,x)$ are determined, we have a family of formal solutions which depends upon an arbitrary function. Such a family will be called a generalized progressing wave.[2] The function $\phi(t,x)$ is called the phase function, the characteristic surfaces ϕ = const. are called phase surfaces, the function $f_o(\phi)$ is called the wave form, and the factors $a^j(t,x)$ are called the distortion or attenuation factors.

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[2] The concept of a progressing wave appears in Courant-Hilbert [1]. F. G. Friedlander [5] observed that the infinite series (1.6) is always a formal solution. Friedlander [6] makes extensive use of generalized progressing waves in treating the wave equation.

Since such progressing wave expansions will not necessarily converge, we must find an interpretation of these expansions which is independent of their convergence. We recall that the geometrical optics expansion (1.1) can be interpreted as a limit of oscillatory solutions as the frequency approaches infinity. The behavior of a function at infinity is closely connected with the smoothness of its Fourier transform. Thus there is a link between highly oscillatory solutions and rapidly varying or discontinuous solutions. If the wave form $f_o(\phi)$ has a discontinuity in one of its derivatives, we observe from the defining relation (1.5) that $f_j(\phi)$ becomes arbitrarily smooth if j is chosen large enough. Thus if we set

$$(1.8) \qquad u^J(t,x) = \sum_{j=0}^{J} f_j(\phi(t,x))\, a^j(t,x) \quad ,$$

then $u^J(t,x)$ differs from an exact solution by a function which becomes arbitrarily smooth (not arbitrarily small) as $J \rightarrow \infty$. This interpretation is valid if we choose an arbitrary distribution for the wave form f_o. Thus the primary usefulness of progressing wave solutions is the following: by means of a finite number of terms in the expansion, we can transform a problem with singular Cauchy data into a problem with arbitrarily smooth Cauchy data. It is noteworthy that such progressing wave expansions are determined by means of ordinary differential equations along the rays.

Because of the great formal similarity between generalized progressing waves and asymptotic expansions, it will be convenient and not misleading to call generalized progressing waves asymptotic solutions.

The phenomenon of caustics furnishes an example of the fruitfulness of the analogy between asymptotic expansions of geometrical optics and generalized progressing waves. Caustics appear where the rays of geometrical optics form an envelope. At a caustic there is a change in the asymptotic behavior of the solution.[3] Thus instead of a leading term of the form

$$v \sim e^{i\omega\phi} a(t,x) \quad , \tag{1.9}$$

we may have

$$v \sim \omega^{\alpha} e^{i\omega\phi} a(t,x) \quad , \tag{1.10}$$

where α is a positive rational number. In physical terms, the intensity of the field becomes infinite at the caustic as the frequency increases. There may also be a shift in phase after a ray passes through the caustic. These phenomena of geometrical optics have their counterparts in the behavior of generalized progressing waves. We shall see that, at a caustic, the wave form is altered. Instead of a leading term of the form

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[3] See I. Kay and J. B. Keller [9].

$$u \sim f_0(\phi)\, a(t,x) \quad , \tag{1.11}$$

we may have

$$u \sim f_{-\alpha}(\phi)\, \hat{a}(t,x) \quad , \tag{1.12}$$

where $f_{-\alpha}$ is an α-th derivative of f_0, and α is a positive rational number. The shift in phase which occurs in optics corresponds to an alteration in the wave form after passing through a caustic.

Although all of the above considerations are valid regardless of whether or not the formal expansion converges, we shall see that, provided that certain analyticity requirements are satisfied, the formal expansions do converge. In particular, we shall see that the Riemann function (a fundamental solution) can be constructed by means of convergent progressing wave expansions, for linear hyperbolic systems with analytic coefficients. In a sense, this result is an extension of the method of Hadamard to more general hyperbolic equations. Since progressing waves can be constructed by means of ordinary differential equations and algebraic processes, hyperbolic equations can thus be solved by means of a recursive system of ordinary differential equations.

C. Examples.

The hyperbolic equation easiest to deal with is the wave equation with two independent variables. All solutions of this equation can be written in the form

$$(1.13) \qquad u(x,t) = F(x+t) + G(x-t) ,$$

where F and G are arbitrary functions of a single variable. Such solutions are called progressing waves. If we prescribe Cauchy data

$$(1.14) \qquad \begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g'(x) , \end{cases}$$

then we can immediately express the solution as

$$(1.15) \qquad u(x,t) = \frac{1}{2} [f(x+t)+g(x+t)+f(x-t)-g(x-t)] .$$

If we do not require that f and g be differentiable in the usual sense, but only that they be distributions, exactly the same formulas give the solution for singular initial data. If $v(x)$ is any C^∞ function with compact support then it follows immediately that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x,t) \, (v_{xx}-v_{tt}) \, dx \, dt = 0 .$$

Thus $u(x,t)$ is a "weak solution" of the wave equation.

However, it is unnecessary to justify our procedure by means of such integrals. On the contrary, if f and g can be represented as limits of twice continuously differentiable functions in some topology which respects addition and multiplication by smooth functions, then our formula (1.15) shows that u is a limit of twice differentiable functions in exactly the same sense. This interpretation of weak solutions is more direct than the usual one, and more meaningful for physical applications. An important result of this paper is that this direct interpretation can be extended to hyperbolic systems of any order, and with any number of independent variables.

Another example of a progressing wave solution is furnished by spherically symmetric solutions of the wave equation with four independent variables. These may be expressed in the form

$$(1.16) \qquad u(r,t) = \frac{F(t+r)-F(t-r)}{r} \quad ,$$

where again F is an arbitrary function. Spherically symmetric solutions of the wave equation with any even number of independent variables have the form[4]

$$(1.17) \qquad u(r,t) = \sum_{j=0}^{(n-4)/2} h^j(r) \, [(-1)^j F_j(t+r) - F_j(t-r)] \, .$$

---

[4] See R. Courant and D. Hilbert [1], p. 415.

Here $F_0(s)$ is an arbitrary function,

$$\frac{d}{ds} F_j(s) = F_{j-1}(s) , \qquad (j = 1, \ldots) ,$$

and the functions $h^j(r)$ are regular for $r > 0$. These solutions are called "progressing waves of higher degree" (höherer stufe). Corresponding to these solutions, weak solutions can be defined and given the same direct interpretation as in the case of two independent variables.

A noteworthy fact about the solution (1.16) is that the differentiability properties of the Cauchy data are preserved in the solution except along the line $r = 0$. This line is the locus of the singularities of the characteristic surfaces $r \pm t = \text{const}$. Such loci of singularities are called "caustics." We shall see that, for any hyperbolic equation in any number of independent variables, differentiability properties of the initial data are preserved in the solution except at caustics of certain characteristic surfaces.

2. Generalized Progressing Wave Solutions of Systems of First Order Equations.

In order to illustrate the method to be followed in the more general case of systems of higher order with multiple characteristics, we shall first construct a family of formal solutions of a system of first order partial differential equations with distinct characteristics. Let $x = (x^o, x^1, \ldots, x^n)$, and let $u(x)$ be the solution vector with k components. Consider the system of k linear first-order equations:

$$\mathcal{L}(u) = A^\nu \frac{\partial u}{\partial x^\nu} + Bu = 0 . \tag{2.1}$$

Here the $k \times k$ matrices A^ν and B are infinitely differentiable, and the index ν is to be summed over from 0 to n whenever it is repeated in a single term. Provided that the planes $x^o = \text{const.}$ are nowhere tangent to a characteristic, we can bring the system (2.1) into a form where the matrix A^o is the identity. We assume that this has been carried out.

We shall construct a formal solution of (2.1) which depends upon an arbitrary function of a single variable: $f_o(s)$. First we define the functions $f_1, f_2, \ldots, f_j, \ldots$ in such a way as to satisfy the conditions

$$\frac{d}{ds} f_j(s) = f_{j-1}(s) \qquad (j = 1,2,\ldots).$$

Then we assume that the solution has the form

$$(2.2) \qquad u(x) = \sum_{j=0}^{\infty} f_j\,(\phi(x))\, a^j(x)\ .$$

f_o is called the wave form, $\phi(x)$ is the phase function, and $a^j(x)$ are the distortion coefficients. Here the coefficients a^j and the function ϕ are to be determined. Next we substitute the expansion (2.2) into the differential equation (2.1):

$$(2.3)\ \mathcal{L}u = f_{-1}(\phi)[A^{\nu}\phi_{\nu}a^{o}] + \sum_{j=0}^{\infty} f_j(\phi)[A^{\nu}\phi_{\nu}a^{j+1}+\mathcal{L}(a^j)] = 0.$$

In (2.3) and henceforth we employ the notation: $\phi_{\nu} = \partial\phi/\partial x^{\nu}$.

In order to determine the coefficients $a^j(x)$, we require that the coefficient of f_j vanish identically for each value of j.[5] Thus we have

$$(2.4) \qquad A^{\nu}\,\phi_{\nu}\,a^{o} = 0\ ,$$

$$(2.5) \qquad A^{\nu}\,\phi_{\nu}\,a^{j+1} + \mathcal{L}(a^j) = 0 \qquad (j = 0,1,\ldots).$$

Hence, from (2.4),

$$(2.6) \qquad \det\,(A^{\nu}\phi_{\nu}) = \Delta(\phi_x) = 0.$$ [6]

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[5] The calculations in the remainder of this section are similar to those in R. Courant and P. D. Lax [4], and identical to those in P. D. Lax [10].

[6] A vector subscript will always denote the gradient with respect to that vector.

Equation (2.6) is the characteristic equation corresponding to the system (2.1). It follows that the phase surfaces $\phi(x) = \text{const.}$ must form a family of characteristic surfaces of equation (2.1).

A second consequence of (2.4) is that a^0 must be a right nullvector of the characteristic matrix $A \equiv A^\nu \phi_\nu$. For convenience, we assume that A has rank $k-1$. This restriction will be removed in Section 4. Then we may define the right nullvector $R(x)$ by the conditions

$$(2.7) \qquad AR = 0 , \qquad |R| = 1 .$$

Consequently a^0 must have the form

$$(2.8) \qquad a^0 = \sigma^0 R ,$$

where σ^0 is a scalar factor still to be determined.

Upon setting $j = 0$ in equation (2.5), we have

$$(2.9) \qquad A a^1 + \mathcal{L}(a^0) = 0 .$$

We may consider (2.9) as a system of linear equations for the vector a^1, where the coefficient matrix A is singular. This system of equations has a solution if and only if

$$(2.10) \qquad L \mathcal{L}(a^0) = 0 .$$

Here L is a left nullvector of A :

$$LA = 0 .$$

Substituting (2.8) into (2.10) yields the following ordinary differential equation for σ^0 :

$$(2.11) \qquad LA^{\nu} R \sigma_{\nu}^{0} + L \mathcal{L}(R) \sigma^{0} = 0 .$$

We shall see presently that the direction of differentiation in (2.11) is bicharacteristic. Provided that (2.11) is satisfied, we can solve (2.9) for a^1, which will then be determined modulo R. Therefore we may write a^1 in terms of a particular solution h^1 and a scalar σ^1 in the form

$$a^1 = h^1 + \sigma^1 R .$$

Now we proceed by induction on j . Let us assume that $a^0,\ldots,a^j$ have been determined in such a way that the coefficients (2.4) and (2.5) of $f_{-1}, f_0, \ldots, f_{j-1}$ are identically zero, and that a^j is determined modulo R:

$$a^j = h^j + \sigma^j R .$$

We set the coefficient of f_j equal to zero (see eq. 2.5):

$$(2.12) \qquad A\, a^{j+1} + \underset{\smile}{\mathcal{L}}(a^j) = 0 .$$

The unknowns in (2.12) are the scalar σ^j and the vector h^{j+1} ($= a^{j+1}$ mod R). The necessary and sufficient condition for the system (2.12) to have a solution is

$$(2.13) \qquad L \mathcal{L}(a^j) = 0 .$$

Hence,

$$(2.14) \qquad L A^{\nu} R \sigma_{\nu}^{j} + L \mathcal{L}(R) \sigma^{j} = -L \mathcal{L}(h^{j}) .$$

This is an inhomogeneous ordinary differential equation for σ^{j}, which has the same left-hand side as (2.11), with σ^{j} in place of σ^{0}. Once σ^{j} has been determined, we can solve (2.12) for h^{j+1}. Thus the induction is complete, and the coefficients a^{j} can be determined successively.

Conversely, suppose that a function ϕ and vectors $a^{0}, a^{1}, \ldots,$ can be determined which satisfy the relations

$$(2.6) \qquad \det (A^{\nu} \phi_{\nu}) = 0 ,$$

$$(2.8) \qquad a^{0} = \sigma^{0} R ,$$

$$(2.12) \qquad A a^{j+1} + \mathcal{L}(a^{j}) = 0 , \quad (j = 0, 1, \ldots).$$

Then our calculations show that $u(x)$, given by (2.2), is a formal solution of the system (2.1). $u(x)$ is uniquely determined if initial values for $\phi(x)$ and $\sigma^{0}(x)$ are assigned.

We now prove

Lemma 2.1: The direction of differentiation in equations (2.11) and (2.14) is bicharacteristic and $\phi(x)$ is constant along a bicharacteristic ray.

The bicharacteristic direction corresponding to a characteristic surface $\phi = \text{const.}$ is given by

$$\frac{dx^\nu}{d\tau} = c(\tau)\,\frac{d\Delta}{\partial\phi_\nu} \qquad (\nu = 0,\ldots,n),$$

where $c(\tau)$ is some constant of proportionality. We can express this ratio in a more convenient form if we determine ϕ_0 as a function of $\phi_1,\ldots,\phi_n$ from eq. (2.6):

$$\phi_0 + \lambda(\phi_1,\ldots,\phi_n) = 0 . \tag{2.15}$$

Note that the function $\lambda(\phi_1,\ldots,\phi_n)$ is homogeneous of degree 1 in its arguments, while $\Delta(\phi_0,\ldots,\phi_n)$ is homogeneous of degree k. Equation (2.15) is more convenient to work with than equation (2.6), since the solution of (2.15) is uniquely determined by its initial values. A solution of (2.6) need not be uniquely determined by its initial values because $\Delta(\phi_0,\phi_1,\ldots,\phi_n)$ is a polynomial of degree k in ϕ_0. A solution of (2.15) is uniquely determined by its initial values because ϕ_0 is given explicitly as a function of $(\phi_1,\ldots,\phi_n)$.

Now the parameter τ may be defined by setting

$$\frac{dx^0}{d\tau} = 1\ , \quad \frac{dx^\mu}{d\tau} = \frac{\partial\lambda}{\partial\phi_\mu} \qquad (\mu = 1,\ldots,n). \tag{2.16}$$

Upon differentiating (2.7) with respect to ϕ_μ, we obtain

$$\left(-\frac{\partial\lambda}{\partial\phi_\mu}A^0 + A^\mu\right)R + A^\nu\phi_\nu\,\frac{\partial R}{\partial\phi_\mu} = 0 .$$

If we multiply this equation by L, we obtain

$$L A^{\mu} R = \frac{\partial\lambda}{\partial\phi_{\mu}} L A^{0} R \quad . \tag{2.17}$$

Upon combining this result with (2.16) we obtain

$$L A^{\nu} R = L A^{0} R \frac{dx^{\nu}}{d\tau} \qquad (\nu = 0,1,\ldots,n). \tag{2.18}$$

Equations (2.18) show that the direction of differentiation in (2.11) and (2.14) is bicharacteristic. Furthermore the homogeneity of the function λ implies that

$$\frac{d\phi}{d\tau} = \phi_0 + \sum_{\mu=1}^{n} \phi_{\mu} \frac{\partial\lambda}{\partial\phi_{\mu}} = \phi_0 + \lambda = 0 \ .$$

Hence ϕ is constant along a bicharacteristic and the lemma is proved.

We may summarize the results of this section in the following

Theorem: $u(x)$, given by (2.2), is a formal solution of (2.1) for arbitrary choice of f_0, provided that

1. the phase function ϕ satisfies the characteristic equation.
2. the scalar factors σ^{j} satisfy the ordinary differential equations (2.11) and (2.14) along the bicharacteristics which generate the surfaces $\phi = \text{const}$.
3. the distortion coefficients a^{j} satisfy the linear algebraic equations (2.12).

3. Generalized Progressing Wave Solutions of the Cauchy Problem for Systems of First Order.

In this section we shall use the formal expansion of Section 2 to construct the formal solution of a Cauchy problem with initial data involving an arbitrary function. We denote x^o by t and $(x^1,\dots,x^n)$ by y. We consider the totally hyperbolic first order system:

$$(3.1) \qquad \mathcal{L}u \equiv A^\nu \frac{\partial u}{\partial x^\nu} + Bu = 0 ,$$

where the $k \times k$ matrices A^ν and B are C^∞, $A^o = I$, and the hyperplanes $t = \text{const.}$ are spacelike: i.e. all real linear combinations of the matrices $A^1,\dots,A^n$ have k real and distinct eigenvalues. The more general case of multiple eigenvalues will be treated in Section 5.

Let us assume that initial data for (3.1) is prescribed in the form of a wave:

$$(3.2) \qquad u(0,y) = \sum_{j=0}^{\infty} f_j(\phi(y))\, b^j(y) ,$$

where

1. f_o is an arbitrary function, and

$$\frac{d}{ds} f_j(s) = f_{j-1}(s) , \qquad (j = 0,1,\dots).$$

2. ϕ is a C^∞ function of y, and the gradient of ϕ with respect to y does not vanish.

3. The vectors $b^j(y)$ are C^∞ functions of y.

First we shall construct k distinct functions[7/] $\phi^a(x)$ which satisfy the characteristic equation

$$\det (A^\nu \phi^a_\nu) = \Delta(\phi^a_\nu) = 0 , \quad (a = 1,\ldots,k), \tag{3.3}$$

with the initial condition

$$\phi^a(0,y) = \phi(y) , \qquad (a = 1,\ldots,k). \tag{3.4}$$

In geometrical terms, we pass k families of characteristic surfaces through the phase surfaces of the initial data. This is equivalent to solving the following set of ordinary differential equations along the rays:

$$\frac{dx^\nu}{d\tau} = \frac{\partial\Delta}{\partial\phi^a_\nu} ; \quad \frac{d\phi^a_\nu}{d\tau} = -\frac{\partial\Delta}{\partial x^\nu} \quad \begin{array}{l}(\nu = 0,1,\ldots,n),\\ (a = 1,\ldots,k),\end{array} \tag{3.5}$$

with the initial conditions

$$\left.\begin{cases} x^\mu(0) = y^\mu , \quad \phi^a_\mu(0) = \dfrac{\partial}{\partial y^\mu}\phi(y) \quad (\mu=1,\ldots,n), \\ x^0(0) = 0 , \quad \Delta(\phi^a_0(0), \phi^a_1(0),\ldots,\phi^a_n(0)) = 0 , \end{cases}\right\} (a=1,\ldots,k) \tag{3.6}$$

The equation

$$\Delta(\phi^a_0(0), \phi^a_1(0), \ldots, \phi^a_n(0)) = 0 ,$$

which appears in (3.6), is the condition that $-\phi^a_0(0)$ be an eigenvalue of the matrix $\phi^a_1(0) A^1 + \ldots + \phi^a_n(0) A^n$.

[7/] This is possible since the plane $t = 0$ is spacelike.

Since we have assumed that the plane $t = 0$ is space-like, there are k distinct roots of the eigenvalue equation. Hence there are k distinct solutions of the system (3.5) with initial condition (3.6), and k distinct solutions of equation (3.3) with the initial condition (3.4). We assume that there is a number $t_1 > 0$ such that the solutions of (3.3) are regular for $0 \leq t \leq t_1$. Such a number always exists if the Cauchy data (3.2) has compact support.

Next we consider a formal expansion $u(x)$ consisting of k expansions, each in the form of a generalized progressing wave. Then we determine the initial values of the coefficients $\sigma^j(x)$ in the expansion from the initial data (3.2). We write

$$u(x) = \sum_{j=0}^{\infty} \sum_{\alpha=1}^{k} f_j(\phi^\alpha(x))\, a^{\alpha j}(x) \; . \tag{3.7}$$

To each of the functions ϕ^α are associated unit vectors R^α and L^α such that

$$\left.\begin{aligned} A^\nu \phi^\alpha_\nu R^\alpha &= 0 \, , \\ L^\alpha A^\nu \phi^\alpha_\nu &= 0 \, . \end{aligned}\right\} \qquad \alpha = 1, \ldots, k. \tag{3.8}$$

We define the scalars $o^{\alpha o}(x)$ by requiring that

$$a^{\alpha o}(x) = \sigma^{\alpha o}(x)\, R^\alpha(x) \, , \qquad (\alpha = 1, \ldots, k). \tag{3.10}$$

Thus, from (3.7),

$$(3.11)\quad u(0,y) = f_o(\phi(y))[\sum_{\alpha=1}^{k} o^{\alpha o}(0,y)R^{\alpha}(0,y)] + + \sum_{j=1}^{\infty} f_j(\phi(y))\ [\ldots]\ .$$

Upon identifying the coefficients of f_o in (3.11) and (3.2) we obtain

$$(3.12)\qquad \sum_{\alpha=1}^{k} \sigma^{\alpha o}\ (0,y)\ R^{\alpha}(0,y) \ = \ b^{o}(y)\ .$$

Since $\phi^1, \phi^2, \ldots, \phi^k$ are distinct, the vectors $R^1, \ldots, R^k$ form a complete set, and the factors $\sigma^{10}, \sigma^{20}, \ldots, \sigma^{ko}$ can be uniquely determined. Once the initial values $\sigma^{\alpha o}(0,y)$ have been determined, we can define the functions $\sigma^{\alpha o}$ along the rays which start at $(0,y)$ by integrating the ordinary differential equations:

$$(2.11)\qquad L^{\alpha}A^{\nu}R^{\alpha}\sigma_{\nu}^{\alpha o} + L^{\alpha}\mathcal{L}(R^{\alpha})\ \sigma^{\alpha o} \ = \ 0 \qquad (\alpha = 1,\ldots,k).$$

Then (2.10) yields $a^{\alpha o}(x)$. This choice of $a^{\alpha o}(x)$ insures the compatibility of the systems

$$(2.9)\qquad A^{\nu}\ \phi_{\nu}^{\alpha}\ a^{\alpha 1} + \mathcal{L}(a^{\alpha o}) \ = \ 0\ , \qquad (\alpha = 1,\ldots,k).$$

Therefore we can determine the vectors $a^{\alpha 1}$ within a multiple of R^{α}:

$$a^{\alpha 1} \ = \ h^{\alpha 1} + \sigma^{\alpha 1}\ R^{\alpha}\ .$$

Returning to the initial conditions, we identify the coefficients of f_1 in (3.2) and (3.11) and obtain

$$\sum_{\alpha=1}^{k} a^{\alpha 1}(0,y) = b^1(y) ,$$

or

$$\sum_{\alpha=1}^{k} \sigma^{\alpha 1} R^\alpha = b^1 - \sum_{\alpha=1}^{k} h^{\alpha 1} . \tag{3.13}$$

In general, we determine $\sigma^{\alpha j}(0,y)$ by solving the system of equations

$$\sum_{\alpha=1}^{k} \sigma^{\alpha j} R^\alpha = b^j - \sum_{\alpha=1}^{k} h^{\alpha j} . \tag{3.14}$$

Then $\sigma^{\alpha j}$ are defined along the rays emanating from $(0,y)$ by integrating the equations

$$L^\alpha A^\nu R^\alpha \sigma_\nu^{\alpha j} + L^\alpha \mathcal{L}(R^\alpha)\, \sigma^{\alpha j} = -L^\alpha \mathcal{L}(h^{\alpha j}), \quad (\alpha=1,\ldots,k). \tag{3.15}$$

$h^{\alpha j+1}$ can be determined by solving the algebraic system

$$A^\nu \phi_\nu^\alpha\, a^{\alpha\, j+1} + \mathcal{L}(a^{\alpha j}) = 0 , \quad (\alpha = 1,\ldots,k). \tag{3.16}$$

Thus we can obtain all of the terms of the series (3.7).

In summary, we recall that the method of Section 2 enables us to construct generalized progressing wave solutions of any linear first order system of partial differential equations with one or more real distinct characteristics. In this section we have considered a totally hyperbolic system and have assigned Cauchy data in the form of a generalized progressing wave. We have shown that if the solution is assumed to consist of a sum of generalized progressing waves, then it has a unique decomposition into a sum of such waves. Each of these waves has a family of characteristic surfaces of the system for phase surfaces. These formal solutions will be discussed in Section 6.

4. Generalized Progressing Wave Solutions of Systems of Higher Order.

In this section we shall extend the method of Section 2 to cover the case of a system of equations (or a single equation) of higher order, where the characteristics need not be distinct.

In order to make ideas clearer and to save writing, it will be convenient to use the following notation, which is due to L. Schwartz. Let

$$x = (x^0, \dots, x^n) = (t, x^1, \dots, x^n) = (t, y) ,$$

$$p = (p_0, \dots, p_n) ,$$

$$D = \left(\frac{\partial}{\partial x^0}, \dots, \frac{\partial}{\partial x^n}\right) ,$$

$$D\phi = \left(\frac{\partial \phi}{\partial x^0}, \dots, \frac{\partial \phi}{\partial x^n}\right) ,$$

$$Q_p = \left(\frac{\partial Q}{\partial p_0}, \dots, \frac{\partial Q}{\partial p_n}\right) ,$$

$$\alpha = (\alpha_0, \dots, \alpha_n) , \quad (\alpha_i \text{ are integers } \geq 0) ,$$

$$p^\alpha = p_0^{\alpha_0} p_1^{\alpha_1} \dots p_n^{\alpha_n} .$$

We shall have to make use of the rule for differentiation of a product of two functions: $\tilde{u}$ and $\hat{u}$. This rule can be expressed in a simple form if we represent D as the sum of the operators $\tilde{D}$ and $\hat{D}$, where D acts only on $\tilde{u}$, and $\hat{D}$ acts only on $\hat{u}$. If $Q(p)$ is a polynomial in

p, then algebraic manipulation shows that

$$Q(\tilde{p}+\hat{p}) = Q(\tilde{p}) + Q_p(\tilde{p})\cdot\hat{p} + \dots ;$$

consequently

$$\begin{aligned} Q(D)(\tilde{u}\,\hat{u}) &= Q(\tilde{D}+\hat{D})\,(\tilde{u}\,\hat{u}) \\ &= Q(\tilde{D})\,\tilde{u}\,\hat{u} + Q_p(\tilde{D})\circ\hat{D}\,\tilde{u}\,\hat{u} + \dots \\ &= (Q(D)\tilde{u})\hat{u} + (Q_p(D)\tilde{u})\circ D\,\hat{u} + \dots . \end{aligned}$$

Now let us consider a $k \times k$ matrix Q whose elements are polynomials in p with coefficients which are C^{∞} functions of x. We write

$$Q(p) = A(p) + B(p) ,$$

where $A(p)$ is homogeneous in p of degree m, and $B(p)$ contains only polynomials of degree less than m. Let u be a vector function of x with k components. We shall be concerned with the linear m^{th} order system of partial differential equations:

$$(4.1) \qquad \mathcal{L}(u) \equiv Q(D)\,u \equiv A(D)u + B(D)u = 0 .$$

We make the following assumptions about the matrix A:

1. If the characteristic polynomial has a root of multiplicity r, then the characteristic matrix has r linearly independent right and left nullvectors.

2. A_{p_o} is proportional to the identity.

3. The multiplicity of each root of the characteristic equation is constant. This condition implies that p_o can be determined as a regular function of $(x, p_1, \ldots, p_n)$ from the characteristic equation.

As before, let $f_o(s)$ be an arbitrary scalar function and let $f_j(s)$ satisfy the relations

$$\frac{d}{ds} f_j(s) = f_{j-1}(s) \qquad (j = -m+1, \ldots).$$

If we first set

$$u(x) = f_o(\phi(x))\, a(x) ,$$

then

$$\mathcal{L} u = f_{-m}(\phi)[A(D\phi)] + f_{-m+1}(\phi)[A_p(D\phi) \cdot Da + Ca] + \ldots,$$

where C is some matrix factor. In order to make our later formulas more explicit we shall calculate C. Since C and all other coefficients in the expansion are independent of f_o, they can be determined easily by a particular choice of f_o. If we set

$$g_m(s, x_o) = \frac{1}{m!} (s - \phi(x_o))^m , \tag{4.2}$$

then

$$\left. \frac{d^j}{d\phi^j} g_m(\phi(x), x_o) \right|_{x=x_o} = \delta_m^j .$$

Thus C can be determined from the condition that

$$(4.3) \qquad Ca\Big|_{x=x_o} = \mathcal{L}(g_{m-1}(\emptyset(x),x_o)a(x_o))\Big|_{x=x_o} .$$

Now if we assume that the solution of (4.1) has the form

$$(4.4) \qquad u(x) = \sum_{j=0}^{\infty} f_j\,(\emptyset(x))\,a^j(x) ,$$

then

$$(4.5) \qquad \mathcal{L}u = f_{-m}(\emptyset)[A(D\emptyset)a^o] +$$

$$+ f_{-m+1}(\emptyset)[A(D\emptyset)a^1+A_p(D\emptyset)\cdot Da^o + Ca^o] +$$

$$+ \sum_{j=1}^{\infty} f_{-m+1+j}(\emptyset)[A(D\emptyset)a^{j+1}+A_p(D\emptyset)\cdot Da^j+Ca^j+K^j],$$

where K^j depends only on a^{j+1-m}, $a^{j+2-m},\ldots,a^{j-1}$, and is in fact given by the formula

$$(4.6) \quad K^j(a^o,\ldots,a^{j-1})\Big|_{x=x_o} = \mathcal{L}[\sum_{\ell=j+1-m}^{j-1} g_{m-1-j+\ell}(\emptyset(x),x_o)a^{\ell}(x)]\Big|_{x=x_o}$$

As before, we determine the function $\emptyset$ and the vectors $a^o,a^1,\ldots$ by setting the coefficient of $f_j(\emptyset)$ equal to zero for $j = -m,\ldots$. Thus we have

$$(4.7) \qquad A(D\emptyset)\,a^o = 0 ,$$

$$(4.8) \qquad A(D\emptyset)a^{j+1} + A_p(D\emptyset)\cdot Da^j+Ca^j+K^j = 0 \quad (j = 0,1,\ldots).$$

Hence, from (4.7)

$$(4.9) \qquad \det (A(D\phi)) \equiv \Delta(D\phi) = 0 .$$

This is the characteristic equation corresponding to the system (4.1). It is a nonlinear first-order partial differential equation for the function $\phi(x)$. Note that $\Delta(D\phi)$ does not explicitly involve the function ϕ, and $\Delta(D\phi)$ is homogeneous of degree mk in $D\phi$. If ϕ satisfies the characteristic equation, then the surfaces $\phi(x) = \text{const.}$ are a family of characteristic surfaces.

If the rank of $A(D\phi)$ is $k-r$, let $R^\beta(x)$ and $L^\gamma(x)$ $(\beta, \gamma = 1,\ldots,r)$ be linearly independent unit right and left nullvectors of $A(D\phi)$.

From equation (4.7), we see that a^o must be a linear combination of $R^1,\ldots,R^r$:

$$(4.10) \qquad a^o = \sigma_1{}^o R^1 + \ldots + \sigma_r{}^o R^r \equiv \sigma_\beta{}^o R^\beta .$$

Setting $j = 0$ in equation (4.8), we have

$$(4.11) \qquad Aa^1 + A_p \cdot Da^o + Ca^o = 0 ,$$

or

$$(4.12) \qquad [Aa^1 + \mathcal{L}(g_{m-1}(\phi,x_o)\, a^o]\Big|_{x=x_o} = 0.$$

This equation has a solution if and only if

$$L^\gamma \mathcal{L}(g_{m-1}(\phi,x_o)a^o)\Big|_{x=x_o} = 0 , \qquad (\gamma = 1,\ldots,r).$$

Hence

$$(4.13)\quad L^{\gamma}A_{p}R^{\beta}\cdot D\sigma_{\beta}^{o} + L^{\gamma}(A_{p}\cdot DR^{\beta} + CR^{\beta})\,\sigma_{\beta}^{o} = 0$$

$$(\gamma = 1,\ldots,r).$$

Now we show that our assumptions about the system (4.1) imply that (4.13) is a system of ordinary differential equations, and the direction of differentiation is bicharacteristic. That is, if we set

$$\rho^{\gamma\beta} = L^{\gamma} A_{p_o} R^{\beta} ,$$

then $L^{\gamma} A_{p_\nu} R^{\beta}$ can be written in the form

$$(4.14)\qquad L^{\gamma} A_{p_\nu} R^{\beta} = \lambda^{(\nu)} \rho^{\gamma\beta} ,$$

for all γ and β, and for $\nu > 0$. It follows from our assumptions 1 and 2 about system (4.1) that $\rho^{\gamma\beta}$ is a nonsingular matrix.

In order to obtain (4.14), we write

$$(4.15)\qquad \phi_o + \lambda(\phi_1,\ldots,\phi_n) = 0 ,$$

in a certain neighborhood of a particular root of (4.9). The bicharacteristic direction and the parameter τ can be defined by

$$(4.16)\qquad \frac{dx^{o}}{d\tau} = 1 , \quad \frac{dx^{\mu}}{d\tau} = \frac{\partial\lambda}{\partial\phi_{\mu}} \qquad (\mu = 1,\ldots,n).$$

λ is homogeneous of degree 1 in $\phi_1,\ldots,\phi_n$; consequently

$$\sum_{\mu=1}^{n} \frac{\partial\lambda}{\partial\phi_{\mu}}\,\phi_{\mu} = \lambda .$$

Hence

$$\frac{d\phi}{d\tau} = \phi_t + \lambda = 0 ,$$

and ϕ is constant along bicharacteristics.

As in the proof of lemma 2.1, we differentiate the equation

$$A(D\phi)\, R^\beta = 0 ,$$

with respect to ϕ_μ and obtain

$$A_{p_o} \frac{\partial\lambda}{\partial\phi_\mu} R^\beta = A_{p_\mu} R^\beta + AR^\beta_{p_\mu} , \qquad (\mu = 1,\ldots,n).$$

Hence

$$L^\gamma A_{p_o} R^\beta \frac{\partial\lambda}{\partial\phi_\mu} = L^\gamma A_{p_\mu} R^\beta , \qquad (\mu = 1,\ldots,n)$$

and from (4.16),

$$(4.17) \qquad L^\gamma A_{p_\nu} R^\beta = (L^\gamma A_{p_o} R^\beta) \frac{dx^\nu}{d\tau} \qquad (\nu = 0,1,\ldots,n),$$

for all β and γ. The relationship expressed by equation (4.17), which greatly simplifies the treatment of multiple characteristics, was first noticed by R. Courant.[8]

It follows from (4.13) and (4.14) that

$$(4.18) \qquad \rho^{\gamma\beta} \frac{d\sigma_\beta{}^o}{d\tau} + P^{\gamma\beta} \sigma_\beta{}^o = 0 \qquad (\gamma = 1,\ldots,r),$$

where

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[8] There is a similar remark by P. Ungar, quoted in R. Lewis [9], p. 577.

$$P^{\gamma\beta}(x_o) = L^{\gamma}\mathscr{L}(g_{m-1}(0,x_o)\, R^{\beta})\Big|_{x=x_o} .$$

Equations (4.18) form a system of ordinary differential equations for $\sigma_1^{\,o},\dots,\sigma_r^{\,o}$, since each term is differentiated in the same direction. If these equations have been satisfied, we can determine a^1 modulo a right nullvector of A from (4.11):

$$a^1 = h^1 + \sigma_{\beta}^{\,1} R^{\beta} .$$

Now we proceed by means of an induction on j. Assume that $a^o,\dots,a^j$ have been determined in such a way that the coefficients of $f_{-m}, f_{-m+1},\dots,f_{j-m}$ in (4.5) are identically zero, and a^j is determined modulo a right nullvector of A:

$$a^j = h^j + \sigma_{\beta}^{\,j} R^{\beta} . \tag{4.19}$$

We set the coefficient of f_{j+1-m} in (4.5) equal to zero:

$$Aa^{j+1} + A_p \circ Da^j + Ca^j + K^j = 0 . \tag{4.20}$$

We determine $\sigma_{\beta}^{\,j}$ by means of the compatibility equations for (4.20):

$$\rho^{\gamma\beta}\frac{d\sigma_{\beta}^{\,j}}{d\tau} + P^{\gamma\beta}\sigma_{\beta}^{\,j} = -L^{\gamma}(A_p \cdot Dh^j + Ch^j + K^j) , \tag{4.21}$$

$$(\gamma = 1,\dots,r).$$

Once $\sigma_{\beta}^{\,j}$ have been determined, we can solve (4.20) for

h^{j+1}. Thus the induction is complete, and the coefficients a^j can be determined in succession.

We conclude that $u(x)$, given by (4.4), is a formal solution of (4.1) for arbitrary choice of f_o, provided that the following conditions are satisfied.

1. The function ϕ satisfies the characteristic equation.
2. The scalar factors σ_β^j satisfy the ordinary differential equations (4.18) and (4.21) along the rays which generate the surfaces ϕ = const.
3. The vectors a^j satisfy the linear algebraic equations (4.20).

5. Generalized Progressing Wave Solutions of the Cauchy Problem for Systems of Higher Order with Multiple Characteristics.

In this section we shall use the formal construction of Section 4 to obtain the solution of a Cauchy problem involving an arbitrary function. We shall see how this solution can be carried out for a class of hyperbolic equations characterized by an algebraic condition.

We label these equations "equations of type H." These are equations of the form

$$\mathcal{L} u \equiv A(D)\, u + B(D)\, u = 0 , \tag{5.1}$$

where the principal part A(D) satisfies:

1. The hyperplanes $x^0 = \text{const.}$ are "spacelike." We interpret this condition as meaning that the determinant of coefficients in equation (5.11) does not vanish.
2. The system (4.13) reduces to ordinary differential equations.[9]
3. The multiplicity of each root of the characteristic equation is constant.
4. All coefficients are C^{∞} functions of x.

The following types of equations are included in class H:

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[9] This condition is satisfied if A_{p_0} is proportional to the identity.

(a) A first-order system, possibly with multiple characteristics, where for fixed $(\emptyset_1,\ldots,\emptyset_n)$, the set of nullvectors $R^{\alpha\beta}$, $(\beta = 1,\ldots,r_\alpha,\ \alpha = 1,\ldots,M)$ of the characteristic matrices $A(D\emptyset^\alpha)$ form a complete set.

(b) A single equation of order m with m real and distinct characteristics for fixed $(\emptyset_1,\ldots,\emptyset_n)$.

As before, we denote x^o by t and $(x^1,\ldots,x^n)$ by y. Consider the Cauchy data

$$(5.2)\quad \begin{cases} u(0,y) = \sum_{j=0}^{\infty} f_j(\emptyset(y))\, b_o^j , \\ u_t(0,y) = \sum_{j=0}^{\infty} f_{j-1}(\emptyset(y))\, b_1^j , \\ \vdots \\ u_{t^{m-1}}(0,y) = \sum_{j=0}^{\infty} f_{j-m+1}(\emptyset(y))\, b_{m-1}^j , \end{cases}$$

where the following additional conditions are satisfied:

5. $f_o(s)$ is an arbitrary function, and

$$\frac{d}{ds} f_j(s) = f_{j-1}(s) \qquad (j = -m+1,\ldots).$$

6. $\emptyset$ is a C^∞ function of y, and the gradient of $\emptyset$ with respect to y does not vanish.

7. The vectors $b_i^j(y)$ are C^∞ functions of y.

As in Section 3, we first construct all possible solutions of the nonlinear first-order partial differential equation:

$$(5.3) \qquad \det\left(A(D\phi^{\alpha})\right) = 0 \qquad (\alpha = 1,\dots,M),$$

with the initial condition

$$(5.4) \qquad \phi^{\alpha}(0,y) = \phi(y) \qquad (\alpha = 1,\dots,M).$$

This is equivalent to solving the associated equations homogeneous of degree 1 (see equation (4.15)):

$$(5.3') \qquad \phi^{\alpha}_{t} + \lambda^{\alpha}(\phi^{\alpha}_{1},\dots,\phi^{\alpha}_{n}) = 0 \qquad (\alpha = 1,\dots,M),$$

or the associated canonical system of ordinary differential equations along the rays:

$$(5.5) \left.\begin{cases} \dfrac{dt}{d\tau} = 1, \quad \dfrac{dx^{\mu}}{d\tau} = \dfrac{\partial\lambda^{\alpha}}{\partial\phi^{\alpha}_{\mu}}, & (\mu = 1,\dots,n) \\[2ex] \dfrac{d\phi^{\alpha}_{\nu}}{d\tau} = -\dfrac{\partial\lambda^{\alpha}}{\partial x^{\nu}} & (\nu = 0,1,\dots,n) \end{cases}\right\} (\alpha=1,\dots,M),$$

with the initial conditions

$$(5.6) \quad \left.\begin{cases} t(0) = 0, \quad x^{\mu}(0) = y^{\mu} \\[1ex] \phi^{\alpha}_{0}(0) = -\lambda^{\alpha}(\phi_{1},\dots,\phi_{n}), \quad \phi^{\alpha}_{\mu}(0) = \phi_{\mu} \end{cases}\right\} (\mu=1,\dots,n).$$

We assume that there is a number $t_1 > 0$ such that the solutions of (5.3) are regular for $0 \le t \le t_1$. Such a number always exists if the Cauchy data (5.2) has compact support.

Using the techniques of Section 4, we can construct a formal solution of the Cauchy problem. We assume that

$w(x)$ has the form:

$$w(x) = \sum_{j=0}^{\infty} \sum_{\alpha=1}^{M} f_j(\phi^\alpha(x))\, a^{\alpha j} . \tag{5.7}$$

To each of the functions $\phi^\alpha(x)$ are associated vectors $R^{\alpha\beta}$ and $L^{\alpha\gamma}$ such that

$$\begin{cases} A(D\phi^\alpha)\, R^{\alpha\beta} = 0 & (\beta = 1,\ldots,r_\alpha;\ \alpha = 1,\ldots,M), \\ L^{\alpha\gamma}\, A(D\phi^\alpha) = 0 & (\gamma = 1,\ldots,r_\alpha;\ \alpha = 1,\ldots,M). \end{cases}$$

We require that $a^{\alpha o}(x)$ have the form

$$a^{\alpha o}(x) = \sum_{\beta=1}^{r_\alpha} \sigma_\beta^{\alpha o}(x)\, R^{\alpha\beta}(x) \equiv \sigma_\beta^{\alpha o}\, R^{\alpha\beta}. \tag{5.9}$$

Then

$$\begin{cases} w(0,y) = f_o(\phi(y))[\sum_{\alpha=1}^{M} \sigma_\beta^{\alpha o} R^{\alpha\beta}] + \sum_{j=1}^{\infty} f_j(\phi)[\ldots], \\ w_t(0,y) = f_{-1}(\phi(y))[\sum_{\alpha=1}^{M} (-\lambda^\alpha)\sigma_\beta^{\alpha o} R^{\alpha\beta}] + \sum_{j=1}^{\infty} f_{j-1}(\phi)[\ldots], \\ \cdot \\ \cdot \\ \cdot \\ w_{t^{m-1}}(0,y) = f_{1-m}(\phi(y))\, [\sum_{\alpha=1}^{M} (-\lambda^\alpha)^{m-1} \sigma_\beta^{\alpha o} R^{\alpha\beta}] + \\ \qquad + \sum_{j=1}^{\infty} f_{j+1-m}(\phi)\, [\ldots] . \end{cases} \tag{5.10}$$

Identifying the first coefficients in equations (5.2) and (5.10), we obtain the system of equations:

$$\tag{5.11} \left\{ \begin{array}{l} \sum_{\alpha=1}^{M} \sigma_\beta^{\alpha o}(0,y)\, R^{\alpha\beta} = b_o^o , \\ \sum_{\alpha=1}^{M} \sigma_\beta^{\alpha o}(-\lambda^\alpha)\, R^{\alpha\beta} = b_1^o , \\ \vdots \\ \sum_{\alpha=1}^{M} \sigma_\beta^{\alpha o}(-\lambda^\alpha)^{m-1}\, R^{\alpha\beta} = b_{n-1}^o . \end{array} \right.$$

This is a system of m vector equations with k components each for the $m\,k$ scalars $\sigma_\beta^{\alpha o}(0,y)$. By hypothesis, the determinant of the system does not vanish. Now we shall verify that this hypothesis is satisfied in cases (a) and (b).

(a) In case (a), $m = 1$. System (5.11) consists of a single vector equation. There are just k vectors $R^{\alpha\beta}$, and by hypothesis they form a complete set. Hence the equations have a unique solution.

(b) In case (b), $k = 1$. System (5.11) consists of m equations for the numbers $\sigma^{\alpha o}$. The determinant of the system is the Van der Monde determinant of the numbers $\lambda^1,\ldots,\lambda^m$. By hypothesis, these numbers are distinct, and hence the determinant does not vanish.

Hence, in each of these cases, the scalars $\sigma_\beta^{\alpha o}$ can be determined from (5.11).

Now the factors $\sigma_\beta^{\alpha o}$ can be determined along the rays which start at $(0,y)$ by integrating the systems of equations:

$$(4.18)\quad \rho^{\alpha\gamma\beta} \frac{d\sigma_\beta^{\alpha o}}{d\tau} + P^{\alpha\gamma\beta} \sigma_\beta^{\alpha o} = 0 \qquad \begin{array}{l}(\gamma = 1,\ldots,r_\alpha)\\ (\alpha = 1,\ldots,M).\end{array}$$

Then (5.9) yields $a^{\alpha o}(x)$. This choice of $a^{\alpha o}$ insures the compatibility of the systems:

$$(4.11)\quad A(D\emptyset^\alpha)a^{\alpha 1}+A_p(D\emptyset^\alpha)\cdot a_x^{\alpha o}+C(\emptyset^\alpha)a^{\alpha o} = 0, \quad (\alpha = 1,\ldots,M),$$

and $a^{\alpha 1}$ can be determined modulo a right nullvector of $A(D\emptyset^\alpha)$:

$$a^{\alpha 1} = h^{\alpha 1} + \sigma_\beta^{\alpha 1} R^{\alpha\beta}, \qquad (A = 1,\ldots,M).$$

Now we can obtain the initial conditions for $o_\beta^{\alpha 1}$ by identifying the coefficients of the second terms in (5.2) and (5.10):

$$(5.12)\quad \begin{cases} \displaystyle\sum_{\alpha=1}^{M} \sigma_\beta^{\alpha 1} R^{\alpha\beta} = b_o^1 - \sum_{\alpha=1}^{M} h^{\alpha 1}, \\ \displaystyle\sum_{\alpha=1}^{M} (-\lambda^\alpha) \sigma_\beta^{\alpha 1} R^{\alpha\beta} = b_1^1 - \sum_{\alpha=1}^{M} \left(\frac{\partial a^{\alpha o}}{\partial t} + (-\lambda^\alpha)h^{\alpha 1}\right), \\ \cdot \\ \cdot \\ \cdot \\ \displaystyle\sum_{\alpha=1}^{M} (-\lambda^\alpha)^{m-1} \sigma_\beta^{\alpha 1} R^{\alpha\beta} = b_{m-1}^1 - \sum_{\alpha=1}^{M}\left[(-\lambda^\alpha)^{m-2} \frac{\partial a^{\alpha o}}{\partial t} + \right. \\ \qquad\qquad \left. + (-\lambda^\alpha)^{m-1} h^{\alpha 1}\right]. \end{cases}$$

In general, the initial conditions for $\sigma_\beta^{\alpha j}$ are determined by solving a system of equations of the form:

$$(5.13)\quad \begin{cases} \displaystyle\sum_{\alpha=1}^{M} \sigma_{\beta}^{\alpha j} R^{\alpha\beta} = b_{o}^{j} - \sum_{\alpha=1}^{M} h^{\alpha j} , \\ \displaystyle\sum_{\alpha=1}^{M} (-\lambda^{\alpha})\, \sigma_{\beta}^{\alpha j} R^{\alpha\beta} = b_{1}^{j} - \sum_{\alpha=1}^{M} (S_{1}^{\alpha j}+(-\lambda^{\alpha})h^{\alpha j}), \\ \quad\vdots \\ \displaystyle\sum_{\alpha=1}^{M} (-\lambda^{\alpha})^{m-1}\sigma_{\beta}^{\alpha j} R^{\alpha\beta} = b_{m-1}^{j} - \sum_{\alpha=1}^{M} (S_{m-1}^{\alpha j}+(-\lambda^{\alpha})^{m-1}h^{\alpha j}), \end{cases}$$

where $S_{\mu}^{\alpha j}$ depends linearly on $a^{\alpha o}, \dots, a^{\alpha\, j-1}$ and their derivatives with respect to t.

The factors $\sigma_{\beta}^{\alpha j}$ can be determined by integrating the systems of equations:

$$(4.21)\quad \rho^{\alpha\gamma\beta} \frac{d\sigma_{\beta}^{\alpha j}}{d\tau} + P^{\alpha\gamma\beta} \sigma_{\beta}^{\alpha j} = -L^{\gamma}(A_{p}\circ h_{x}^{j} + Ch^{j} + K^{j}),$$

$$(\gamma=1,\dots,r_{\alpha}),\ (\alpha=1,\dots,M).$$

Then the equations

$$(4.20)\quad A(D\phi^{\alpha})\, a^{\alpha\, j+1} + A_{p}\circ Da^{\alpha j} + Ca^{\alpha j} + K^{\alpha j} = 0 ,$$

can be solved for $h^{\alpha\, j+1}$. Thus the terms of the series (5.6) can be constructed in succession.

The argument of this section has shown that if we assign Cauchy data in the form of a wave to a hyperbolic system of type H, and if the formal solution is assumed to consist of a sum of generalized progressing waves, then the formal solution has a unique decomposition into a sum of such waves. Each of these waves has a family of characteristic surfaces of the system for phase surfaces. We shall apply this formal construction in the next section.

6. Applications of the Formal Construction

A. Progressing waves and progressing waves of higher degree.

It can happen that, by a fortunate combination of circumstances, all but a finite number of the distortion coefficients $a^{\alpha j}$ vanish.[10] In this case the formal solution is an actual solution, or rather a class of actual solutions, since the wave form $f_o(s)$ may be chosen arbitrarily. Such a class of solutions is called a progressing wave of higher degree. If only the first terms $a^{\alpha o}$ are different from zero, the solutions are called progressing waves. Examples of progressing waves and of progressing waves of higher degree were given in the introduction.

B. Power series expansions.

If we choose $f_j(s) = \frac{1}{j!} s^j$, then our formal expansion becomes a power series in the functions $\phi^\alpha(x)$. In many cases this series converges (see Section 7), and thus represents an actual solution of the Cauchy problem. In this case our method of solution is similar to that used by J. Hadamard [5] to construct the fundamental solution for equations of second order.

C. Asymptotic expansions.

If we choose $f_j(s) = (i\omega)^{-j} e^{i\omega s}$, then our formal expansion becomes an asymptotic expansion of the solution of the Cauchy problem in terms of the frequency parameter ω. In this case our method of solution of the Cauchy problem is identical with that used by P. D. Lax [10] to obtain such asymptotic expansions for solutions of systems of first-order equations.

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[10] Such a situation has been analyzed for the wave equation by F. G. Friedlander [6].

D. Weak solutions of problems with distribution initial data.

Since the distortion factors $a^{\alpha j}$ do not depend upon the wave form $f_o(s)$ involved in the initial data, we can allow generalized functions f_o, which are limits of smooth functions in a topology which respects addition and multiplication by a smooth function. In particular, if f_o is a distribution, then the corresponding formal solution of the Cauchy problem enables us to construct a weak solution of the Cauchy problem.

More specifically, let $f_o(s) = d^P/ds^P\, g(s)$ where $g(s)$ is merely a continuous function, and P is a positive integer. We assume further that only a finite number of the factors b_i^j are different from zero (see eq. (5.2)). Let

$$u^J(x) = \sum_{j=0}^{J} \sum_{\alpha=1}^{M} f_j(\phi^\alpha(x))\, a^{\alpha j}(x) \ .$$

The factors $a^{\alpha j}$ have been constructed in such a way that

$$\begin{aligned} \mathcal{L}(u^J) &= \sum_{\alpha=1}^{M} \Big\{ f_{J-m+1}(\phi^\alpha)[A_p \cdot Da^{\alpha J} + Ca^{\alpha J} + K^{\alpha J}] + \sum_{\ell=J-m+2}^{J} f_\ell(\phi^\alpha)[\ldots] \Big\}. \\ &= \sum_{\alpha=1}^{M} \Big\{ g_{J-P-m+1}(\phi^\alpha)[A_p \cdot Da^{\alpha J} + Ca^{\alpha J} + K^{\alpha J}] + \sum_{\ell=J-P-m+2}^{J-P} g_\ell(\phi^\alpha)[\ldots] \Big\}. \end{aligned}$$

Thus $\mathcal{L}(u^J)$ has $J-P-m+1$ derivatives in the ordinary sense. We now define a function $v(x)$ by the conditions

$$(6.1) \qquad \mathcal{L}(v) = -\mathcal{L}(u^J) ,$$

$$(6.2) \qquad v(0,y) = v_t(0,y) = \dots v_{t^{m-1}}(0,y) = 0 .$$

This is a Cauchy problem and it has smooth data provided that J is taken sufficiently large. Therefore the standard existence and uniqueness theory (see Leray [11]) implies that v exists, is uniquely determined, and has as many derivatives as we like if J is chosen sufficiently large.[11] Now we define $u(x)$ by

$$u(x) = u^J(x) + v(x) .$$

This function $u(x)$ is the unique solution of the Cauchy problem (5.1), (5.2).

We can summarize our results in the following

Theorem: If the singular part of the Cauchy data has the form of a wave (5.2), and if there are no caustics (which is the case for $t \leq t_1$), then

1. The singular part of the solution has the form of a sum of generalized progressing waves. If $\phi(y)$ is the phase function for the initial data, then the phase

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[11] Once J has been chosen, the differentiability conditions on the coefficients of the equations and on the initial data can be relaxed. We shall not attempt to show how many derivatives are actually required.

functions for the singular part of the solution are solutions of the characteristic equation of the system which agree with $\phi(y)$ for $t = 0$. Thus if the family of curves $C(c)$ is defined by the equation $\phi(y) = c$, then the solution is a smooth function except along the characteristic surfaces $S^{\alpha}(c)$ which emanate from $C(c)$.

2. The distribution $f_o(s)$ which gives the wave form of the data along $C(c)$ also gives the wave form of the solution along $S^{\alpha}(c)$. Consequently

3. The solution has the same smoothness properties as the initial data.

4. The distortion coefficients $a^{\alpha j}$ are determined by solving linear first-order ordinary differential equations along the rays which generate the characteristic surfaces $S^{\alpha}(c)$.

Moreover,

5. These coefficients are independent of the particular wave form in the initial data.

6. The coefficients $a^{\alpha j}$ are identical with coefficients which appear in the asymptotic expansion of the solution of a Cauchy problem with rapidly oscillating initial data.

7. Proof of Convergence in the Analytic Case.

We have pointed out in the introduction that progressing wave expansions are meaningful regardless of whether or not they are convergent. Nevertheless, the situation is simpler when such expansions converge. It is apparent that convergence of such expansions must be connected with analyticity of the equation and the data, since a power series expansion can be obtained from a generalized progressing wave by appropriate choice of the wave form. In fact, we can show that if the coefficients of the equation are analytic, and if the distortion coefficients of the initial data and the initial values of the phase function are analytic, and if the power series corresponding to the initial values of the wave is convergent, then the wave converges in an appropriate neighborhood of the initial surface. Note that no assumptions need be made about the wave form. In particular, the theorem applies to progressing wave expansions where the wave form is a distribution. The theorem also applies to equations which are not hyperbolic, except that for non-hyperbolic equations the Cauchy data cannot be chosen arbitrarily. The proof consists in transforming our problem into an analytic characteristic initial value problem, and applying an extension of the Cauchy-Kowalewski theorem. For simplicity, we present the proof for a first-order system.

First we shall give an extension of the Cauchy-Kowalewski theorem to apply to the characteristic initial value problem.[12] In order to clarify the presentation, we shall first prove a theorem which is weaker than the one we need.

Theorem 7.1. Let $A^\nu(x)$ $(\nu = 0,\ldots,n)$, $B(x)$ and $C(x)$ be $k \times k$ matrices whose elements are analytic functions of $(x^0,\ldots,x^n)$ in a domain D in (z,x)-space which includes the origin. Let $C(x)$ have rank $k-1$, and let $R(x)$ and $L(x)$ be unit right and left nullvectors of $C(x)$ respectively. We assume that the scalar $LA^0R \neq 0$. We may also assume without loss of generality that $\sum_{\nu=0}^{n} LA^\nu R > 0$ in D. Let $b(z,x^1,\ldots,x^n)$ be a vector with k components, each of which is analytic in D, with $b(0,x^1,\ldots,x^n) = 0$.

Consider the system of k linear first-order partial differential equations

$$(7.1) \qquad C(x)\frac{\partial u(z,x)}{\partial z} = \sum_{\nu=0}^{n} A^\nu(x)\frac{\partial u(z,x)}{\partial x^\nu} + B(x)\,u(z,x),$$

with the subsidiary conditions

$$(7.2) \qquad u(0,x) = 0,$$

$$(7.3) \qquad [u(z,0,x^1,\ldots,x^n) - b(z,x^1,\ldots,x^n)]\cdot R(x) = 0.$$

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[12] As far as I know, such a theorem has been proved only for second-order equations (see Hadamard [7] pp. 77-81).

If the above hypotheses are satisfied, then (7.1), (7.2) and (7.3) have a unique analytic solution in an appropriate neighborhood of the origin.

Proof: First we remark that (7.1) and (7.2) constitute a characteristic Cauchy problem. Now we shall show that equation (7.3) removes the indetermination in the formal expansion of the solution.

It follows from (7.1) that

$$L\left(\sum_{\nu=0}^{n} A^{\nu} \frac{\partial u}{\partial x^{\nu}} + Bu \right) = 0 . \tag{7.4}$$

If (7.4) is satisfied, then $\partial u/\partial z$ is determined in terms of u and $\partial u/\partial x^{\nu}$ modulo a right nullvector of C. We write

$$u(z,x) = h(z,x) + \sigma(z,x)\, R(x) , \tag{7.5}$$

where σ is a scalar factor. From (7.1), we may obtain

$$h_z = \mathcal{L}_1(h) + \mathcal{L}_2(\sigma) , \tag{7.6}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are first-order partial differential operators which involve only the x-variables. Equation (7.4) may be rewritten in the form

$$\frac{d\sigma}{d\tau} + P(x)\, \sigma = \mathcal{L}_3(h) , \tag{7.7}$$

where $dx^{\nu}/d\tau = LA^{\nu}R$, $P(x) = \sum_{\nu=0}^{n} LA^{\nu}R_{\nu}$, and $\mathcal{L}_3$ is a first-order partial differential operator which involves only the x-variables. Equation (7.3) may be rewritten in the form

$$(7.8) \qquad \sigma(z,0,x^1,\ldots,x^n) = (b-h)\cdot R .$$

The formal expansion of u can be computed from (7.6), (7.7) and (7.8). It remains to show that this formal expansion converges in a neighborhood of the origin. To do this, we shall construct majorants for h and σ.

Let $y = \sum_{\nu=0}^{n} x^\nu$. Let $M_1/(1-y/\rho_1)$ majorize the norms of the coefficients of $\mathcal{L}_1$ and $\mathcal{L}_2$. If S majorizes σ, and if H satisfies

$$(7.9) \qquad H_z = \frac{M_1}{1-y/\rho_1}\left(\sum_{\nu=0}^{n} \frac{\partial H}{\partial x^\nu} + H + \sum_{\nu=0}^{n} \frac{\partial S}{\partial x^\nu} + S\right) ,$$

$$(7.10) \qquad H(0,x) = 0 ,$$

then $|h| < H$. In other words, if we have a majorant for σ, then we can easily obtain a majorant for h. Note that in order to estimate the coefficients of z^ℓ in the expansion for h, it is only necessary to estimate the coefficients of $z^1,\ldots,z^{\ell-1}$ in the expansion of σ. Note also that if S depends only upon z and y, then H depends only upon z and y.

Now we proceed to estimate σ in terms of h. From (7.7) and (7.8), we may represent σ in the form

$$(7.11)\qquad \sigma(z,x) = \int k(x,\tau)\, \mathcal{L}_3[h(z,x(\tau))]\, d\tau + {} + m(x)\,[b(z,x') - h(z,x')] .$$

Here $k(x,\tau)$ and $m(x)$ are bounded functions, and x' is the intersection of the ray containing x with the surface $x^0 = 0$. Since the vectors $(LA^0R,\ldots,LA^nR)$ and $(1,\ldots,1)$ are not orthogonal (see the hypothesis) we may introduce $y'' = \sum_{\nu=0}^{n} x^{\nu}(\tau)$ as variable of integration instead of τ. It follows that if $H(z,x)$ is a majorant for h, H is a function of z and y only, and M_i, ρ_i are properly chosen, then

$$(7.12)\quad |\sigma| < \int \left[\frac{M_2}{1-\frac{y+y''}{\rho_2}} \Big(\sum_{\nu=0}^{n} H_\nu\Big) + \frac{M_3}{1-\frac{y+y''}{\rho_3}} H \right] dy'' + {} + \frac{M_4}{1-\frac{y}{\rho_4}} H(z,x') + \frac{zM_5}{(1-\frac{z}{\rho_5})(1-\frac{y}{\rho_6})} .$$

But $\sum_{\nu=0}^{n} H_\nu = (n+1)\frac{d}{dy} H$. Hence, integrating by parts,

$$(7.13)\quad |\sigma| < - \int \left[(n+1)M_2 \frac{\partial}{\partial y''}\Big(1- \frac{1}{\frac{y+y''}{\rho_2}}\Big) - \frac{M_3}{1-\frac{y+y''}{\rho_3}} \right] H(z,x(y''))dy'' + {} + \frac{M_2(n+1)}{1-\frac{2y}{\rho_2}} H(z,x) - \frac{M_2(n+1)}{1-\frac{y+y'(y)}{\rho_2}} H(z,x') + \frac{M_4}{1-\frac{y}{\rho_4}} H(z,x') + {} + \frac{zM_5}{(1-\frac{z}{\rho_5})(1-\frac{z}{\rho_6})} .$$

If M_2 is chosen large enough, and ρ_2 is chosen small enough, then the integrand in (7.13) can be made positive, and the term involving $H(z,x')$ can be made negative. Hence the inequality is improved if these terms are dropped. Thus we obtain an inequality of the form

$$(7.14) \qquad |\sigma| < \frac{M_7}{1-\frac{y}{\rho_7}} H(z,x) + \frac{zM_7}{(1-\frac{z}{\rho_7})(1-\frac{y}{\rho_7})} \, .$$

We may set

$$(7.15) \qquad S = \frac{M_7}{1-\frac{y}{\rho_7}} H(z,x) + \frac{zM_7}{(1-\frac{z}{\rho_7})(1-\frac{y}{\rho_7})} \, .$$

Note that the coefficient of z^{ℓ} in the expansion of σ may be estimated in terms of the coefficient of z^{ℓ} in the expansion of H. Also, if H depends only on z and y, then S depends only on z and y.

We assert that a majorant of h is given by an equation of the form

$$(7.16) \qquad H_z = \frac{M}{1-\frac{y}{\rho}} \left[\sum_{\nu=0}^{n} \frac{\partial H}{\partial x^{\nu}} + H + \frac{z}{1-\frac{z}{\rho}} \right] ,$$

with

$$(7.17) \qquad H(0,x) = 0 \, .$$

A majorant for σ then is given by (7.15). The constants M and ρ are obtained by a formal substitution of (7.15) into (7.9). The proof of this assertion proceeds by means of an induction on ℓ.

Assume that the expansions

$$H^{(\ell-1)} = \sum_{j=1}^{\ell-1} H_j(x)\, z^j ,$$

and

$$S^{(\ell-1)} = \sum_{j=1}^{\ell-1} S_j(x)\, z^j ,$$

majorize the first $\ell-1$ terms in the expansion of h and σ. Assume further that $H^{(\ell-1)}$ was obtained from (7.16) and $S^{(\ell-1)}$ was then obtained from (7.15). We will show that the corresponding statements are true of $H^{(\ell)}$ and $S^{(\ell)}$. It follows from the remark following (7.9) that $H^{(\ell)}$ may be obtained by substituting (7.15) into (7.9), i.e. $H^{(\ell)}$ is given by (7.16). In a similar way, we see that $S^{(\ell)}$ is given by (7.15). It only remains to show that the induction can be started. This is indeed the case, since $H^{(1)} = 0$ and $S^{(1)} = M_7 z$.

Since (7.16), (7.17) obviously have a unique solution, we conclude that the expansions of h and σ converge in a sufficiently small neighborhood of the origin.

As we have mentioned, Theorem 7.1 is not strong enough for our purposes: we shall have to deal with the case where the initial manifold is multiply characteristic. In order to avoid clumsy algebraic hypotheses, we shall assume that our equations are already in a form corresponding to (7.6), (7.7) and (7.8).

Theorem 7.2: Consider the set of equations

$$(7.18) \qquad h_z = \mathcal{L}_1(h) + \mathcal{L}_2(\sigma^1, \ldots, \sigma^2) ,$$

$$(7.19) \qquad \frac{d\sigma^\alpha}{d\tau^{(\alpha)}} + P^\alpha(x)\, \sigma^\alpha = \mathcal{L}_3^\alpha(h) \qquad (\alpha = 1, \ldots, r),$$

$$(7.20) \qquad \sigma^\alpha(z, 0, x^1, \ldots, x^n) = E^\alpha(x) h + F^\alpha(x) b , \quad (\alpha = 1, \ldots, r).$$

Here h is a vector, and σ^α may be either a vector or a scalar. $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3^\alpha$ are first-order differential operators which involve only the x-variables. $P^\alpha(x)$, $E^\alpha(x)$ and $F^\alpha(x)$ are coefficients which may be scalars, vectors or matrices, depending upon whether σ^α is a scalar or a vector. All functions are assumed to be analytic in a neighborhood of the origin of (z,x)-space. We assume further that $\frac{dx^0}{d\tau^{(\alpha)}} \neq 0$. By means of an affine transformation, we may insure that $y = \sum_{\nu=0}^{n} x^\nu$ may be introduced as a curve parameter in (7.19) instead of $\tau^{(\alpha)}$. If all of the above hypotheses are satisfied, then equations (7.18), (7.19), and (7.20) (and of course the corresponding Cauchy problem) have a unique analytic solution .

The proof is an obvious modification of the proof of Theorem 7.1, and will be omitted.

After these preliminaries, we can easily prove

Theorem 7.3: If $\mathcal{L}u = 0$ is a system of linear first-order hyperbolic equations with analytic coefficients of type H (see Section 5), and Cauchy data is prescribed on the spacelike manifold $t = 0$ in $(t, x^1, \ldots, x^n)$ space in the form

$$(7.21) \qquad u(0,x) = \sum_{j=0}^{\infty} b^j(x)\, f_j(\phi(x)) \, ,$$

where

1. f_0 is the m-th derivative of a continuous function,
2. $\frac{d}{d\phi} f_j(\phi) = f_{j-1}(\phi)$,
3. $f_j(0) = 0$ if $j > m$,
4. $\phi(x)$ is analytic, and grad $\phi \neq 0$,
5. $b^j(x)$ are analytic, and the series $\sum_{j=0}^{\infty} b^j(x) \frac{z^j}{j!}$ converges,

then the solution has the form

$$(7.22) \qquad u(t,x) = \sum_{\alpha=1}^{k} \sum_{j=0}^{\infty} a^{\alpha j}(t,x)\, f_j(\phi^\alpha(t,x))$$

where $\phi^\alpha(t,x)$ are the k solutions of the characteristic equation, with $\phi^\alpha(0,x) = \phi(x)$, and $a^{\alpha j}(t,x)$ are coefficients which are independent of $f_0(\phi)$.

Proof: We first remark that it is sufficient to prove the convergence of the series

$$(7.23) \qquad v(t,x) = \sum_{\alpha=1}^{k} \sum_{j=0}^{\infty} \frac{(\phi^\alpha)^j}{j!}\, a^{\alpha j}(t,x) \, ,$$

since

(7.24) $$|f_j(\emptyset)| < M \frac{|\emptyset|^{j-m}}{(j-m)!}, \qquad \text{for} \quad j > m .$$

$a^{\alpha j}$ are determined from the following set of equations:

(7.25) $$A^{(\alpha)} a^{\alpha j+1} = -\mathcal{L}(a^{\alpha j}) \qquad (j = -1,\ldots), \quad (\alpha = 1,\ldots,k),$$

where $A^{(\alpha)} = A^{\nu}\emptyset^{\alpha}_{\nu}$. $a^{\alpha j}$ must also satisfy the initial condition

(7.26) $$\sum_{\alpha=1}^{k} a^{\alpha j}(0,x) = b^{j} \qquad (j = 0,1,\ldots).$$

In order to deal with this infinite set of equations, we introduce the generating functions $g^{\alpha}(z;t,x)$ and $b(z,x)$:

(7.27) $$g^{\alpha}(z;t,x) = \sum_{j=0}^{\infty} a^{\alpha j}(t,x) \frac{z^{j+1}}{(j+1)!},$$

(7.28) $$b(z;x) = \sum_{j=0}^{\infty} b^{j}(x) \frac{z^{j+1}}{(j+1)!}.$$

The infinite system (7.25) is equivalent to the set of k partial differential equations:

(7.29) $$A^{(\alpha)} \frac{\partial g^{\alpha}}{\partial z} = -\mathcal{L}(g^{\alpha}) \qquad (\alpha = 1,\ldots,k),$$

and the initial conditions become

(7.30) $$\sum_{\alpha=1}^{k} g^{\alpha}(z;0,x) = b(z,x).$$

By definition, g^{α} satisfies the condition

(7.31) $$g^{\alpha}(0;t,x) = 0 \qquad (\alpha = 1,\ldots,k).$$

For fixed α, equations (7.29) and (7.31) constitute a Cauchy problem for $g(z;t,x)$, where the initial manifold is given by $z = 0$, not $t = 0$. Since ϕ^α satisfies the characteristic equation, the matrix $A^{(\alpha)}$ has a vanishing determinant, and hence the manifold $z = 0$ is characteristic. Hence $g^\alpha(z;t,x)$ need not vanish identically, even though it must satisfy a homogeneous equation and assume zero initial values (for $z = 0$). Indeed, g^α is not uniquely determined unless conditions are prescribed on another surface, in this case the manifold $t = 0$. This additional condition (7.30) links the systems (7.29).

In order to treat condition (7.30), it is expedient to write our equations in still another form. Let G be a vector with k^2 components:

$$G = \begin{pmatrix} g^1 \\ \cdot \\ \cdot \\ \cdot \\ g^k \end{pmatrix} ,$$

let C be the k^2 x k^2 matrix

$$C = \begin{pmatrix} A^{(1)} & & 0 \\ & \ddots & \\ 0 & & A^{(k)} \end{pmatrix} .$$

We have

(7.32) $$C \frac{\partial G}{\partial z} = - \mathcal{L}(G) ,$$

(7.33) $$G(0;t,x) = 0 ,$$

(7.34) $$\sum_{\alpha=1}^{k} g^{\alpha}(z;0,x) = b(z;x) .$$

This is another characteristic inital value problem, but in this case the manifold $z = 0$ is characteristic with multiplicity k. Equation (7.34) gives just k additional conditions on the manifold $t = 0$. The hypotheses of the theorem imply that the coefficients in equation (7.29) and the data (7.34) are analytic.

Since our equations are of type (H), the coefficients $a^{\alpha j}$ are determined from equations (4.20) and (4.21). These equations correspond exactly to (7.18) and (7.19), while our condition (7.30) immediately yields an equation corresponding to (7.20). Thus we can apply Theorem 7.2, and the series (7.23) for $v(t,x)$ converges if $\phi^1,\dots,\phi^k, t,$ and x are sufficiently small.

8. The Riemann Function.

In this section we shall give a method for constructing the Riemann function for equations of type H.

We may summarize our results as follows:

1. The singular part of the Riemann function can be represented as a superposition of waves whose phase surfaces are regular characteristic surfaces.

2. In the analytic case, the Riemann function can be represented as a superposition of convergent progressing waves. These convergent progressing waves are constructed in a manner similar to Hadamard's method for equations of second order.

3. The Riemann function is a smooth function except if its arguments lie on the same ray.

A. Differential operators and their adjoints.

We shall make use of the notation of Section 4. Let

$$Q(p) = \sum_{\alpha} q_{\alpha}(x)\, p^{\alpha} , \tag{8.1}$$

where $q_{\alpha}(x)$ are $k \times k$ matrices, $q_{\alpha} \equiv 0$ if $|\alpha| > m$ and $q_{m,0,0,\ldots,0} = I$. Let

$$u \equiv Q(D)u \equiv \sum_{\alpha} q_{\alpha}(x)\, D^{\alpha} u . \tag{8.2}$$

We define $\mathcal{L}^* v$ by

$$\mathcal{L}^* v \equiv \sum_{\alpha} (-D)^{\alpha}\, v\, q_{\alpha} . \tag{8.3}$$

$\mathcal{L}^*$ is called the formal adjoint to $\mathcal{L}$. $\mathcal{L}$ and $\mathcal{L}^*$ satisfy the relation

$$(8.4) \quad v\mathcal{L}u - (\mathcal{L}^*v)u = \sum_{i=0}^{n} D_i \, P^i(v,u) \, ,$$

where $P^i(v,u)$ is a bilinear differential operator whose coefficients depend on $q_\alpha(x)$.

The right-hand side of (8.4) is clearly a divergence. Let V be a region of x-space with boundary S. Gauss' theorem implies that

$$(8.5) \quad \int_V [v\mathcal{L}u-(\mathcal{L}^*v)u] \, dx = \int_S \sum_{i=0}^{n} P^i(v,u) \, dS_i \, .$$

This is the fundamental relation between a differential operator and its adjoint, sometimes called Green's identity. If $v(\tau_2,y) = v_t(\tau_2,y) = \dots = v_{t^{m-2}}(\tau_2,y) = 0$, and v has compact support in every plane $t = \text{const.}$, then, integrating (8.4) over the region $\tau_1 \leq t \leq \tau_2$, we have

$$(8.6) \quad \int_{\tau_1\leq t\leq\tau_2} [v\mathcal{L}u-(\mathcal{L}^*v)u]dx = \int (-1)^{m-1} v_{t^{m-1}}(\tau_2,y)u(\tau_2,y)dy - \int P^0[v(\tau_1,y),u(\tau_1,y)] \, dy,$$

since $q_{m,0,\dots,0} = I$.

B. The Riemann function and its construction.

To any hyperbolic system, we can associate a fundamental solution which bears a particularly useful relationship to the Cauchy problem, namely the Riemann function. The Riemann function corresponding to the operator $\mathcal{L}$ satisfies the following conditions:

$$(8.7)\quad \begin{cases} 1.\quad \mathcal{L}^{*}k(t,y,\tau,\eta) = 0 & \text{if} \quad t < \tau, \\ 2.\quad k(\tau,y,\tau,\eta) = k_t(\tau,y,\tau,\eta) = \ldots = k_{t^{m-2}}(\tau,y,\tau,\eta)=0, & \\ 3.\quad k_{t^{m-1}}(\tau,y,\tau,\eta) = (-1)^{m-1}\delta(y-\eta)I, & \\ 4.\quad k(t,y,\tau,\eta) = 0 & \text{if} \quad t > \tau. \end{cases}$$

$\delta(y)$ denotes, of course, the n-dimensional Dirac distribution. The uniqueness theory for hyperbolic equations implies that k has compact support in each plane $t = \text{const}$. Inserting (8.7) into (8.6), we have

$$(8.8)\quad u(\tau_2,\eta) = \int_{\tau_1\leq t\leq \tau_2} k(t,y,\tau_2,\eta)\,\mathcal{L}u \,dt\,dy + \int P^{0}[k(\tau_1,y,\tau_2,\eta)\,,\,u(\tau_1,y)]\,dy\,.$$

It is because of this formula, which expresses any function $u(x)$ in terms of $\mathcal{L}u$ and the Cauchy data of u on the hyperplane $t = \tau_1$, that the Riemann function is so important.

We may define the Riemann function for the operator $\mathcal{L}^{*}$ as follows:

$$(8.9)\begin{cases} 1.\ \mathcal{L}k^*(t,y,\tau,\eta) = 0 & \text{if } t > \tau, \\ 2.\ k^*(\tau,y,\tau,\eta) = k^*_t(\tau,y,\tau,\eta) = \dots = k^*_{t^{m-2}}(\tau,y,\tau,\eta) = 0, & \\ 3.\ k^*_{t^{m-1}}(\tau,y,\tau,\eta) = \delta(y-\eta)\, I, & \\ 4.\ k(t,y,\tau,\eta) = 0 & \text{if } t < \tau. \end{cases}$$

We set $v(t,y) = k(t,y,\tau_2,\eta_2)$, $u(t,y) = k^*(t,y,\tau_1,\eta_1)$, and substitute into (8.6). This yields immediately that

$$(8.10) \qquad k^*(\tau_2,\eta_2,\tau_1,\eta_1) = k(\tau_1,\eta_1,\tau_2,\eta_2) .$$

This symmetry relation exhibits the close relationship between a system of equations and its adjoint, and demonstrates that the function $k(t,y,\tau,\eta)$ satisfies

$$(8.11) \qquad \mathcal{L}_{\tau,\eta} k(t,x,\tau,\eta) = 0 , \qquad (\tau > t)$$

with respect to the variables τ and η.

The Riemann function can be constructed by means of progressing waves, provided that the δ-function can be represented as a superposition of waves with regular phase surfaces, for example plane waves.[13] Such a representation is given in F. John [8], p. 11:

$$\delta(y) = - \frac{\Delta^{(n+k)/2}}{(2\pi i)^n} \int_{|\omega|=1} (y\cdot\omega)^k \log y\cdot\omega \; d\omega,$$

where $\log y\cdot\omega$ denotes the complex-valued function defined on the real axis. We choose $k = 1$ if n is odd, and $k = 0$ if u is even, and differentiate under the integral sign. This immediately yields the formula[14]

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[13] This method was devised by R. Courant and P. D. Lax [4].

$$(8.12) \qquad \delta(y) = -\frac{1}{(2\pi i)^n} \int_{|\omega|=1} \log_{-n} y \cdot \omega \, d\omega ,$$

where $\log_{-n}$ denotes the n-th derivative of the log, in harmony with our previous notation.

Let $w(t, y, \tau, \eta, \omega)$ satisfy the following conditions:

$$(8.13) \begin{cases} 1. \ \mathcal{L}^* w = 0 & \text{if } t < \tau , \\ 2. \ w(\tau, y, \tau, \eta, \omega) = w_t(\tau, y, \tau, \eta, \omega) = \dots = \\ \qquad = w_{t^{m-2}}(\tau, y, \tau, \eta, \omega) = 0 , & \\ 3. \ w_{t^{m-1}}(\tau, y, \tau, \eta, \omega) = -\frac{1}{(2\pi i)^n} \log_{-n}((y - \eta) \cdot \omega) & \\ 4. \ w(t, y, \tau, \eta, \omega) = 0 & \text{if } t > \tau . \end{cases}$$

From the principle of superposition, we have

$$(8.14) \qquad k(t, y, \tau, \eta) = \int_{|\omega|=1} w(t, y, \tau, \eta, \omega) \, d\omega .$$

w can be constructed by means of progessing waves, provided that the system $\mathcal{L}^* w = 0$ satisfies the algebraic conditions listed in Section 5. Moreover, if the coefficients of $\mathcal{L}$ (and hence of $\mathcal{L}^*$ are analytic, then the formal series expansion for w converges in a neighborhood of the initial surface. Since every solution of the Cauchy problem can be represented in terms of k, we are led to a solution of the Cauchy problem by repeated application of differentiation, integration and algebraic processes. We can obtain approximate solutions after a finite number of such operations by breaking off the series after a finite number of terms. This is, in a sense, a generalization of the method and results of Hadamard to general analytic hyperbolic equations.

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[14] This formula was brought to my attention by L. Gårding.

C. The generalized Huygens principle.

The construction by means of progressing waves shows that $w(x,\xi,\omega)$ may be written in the form

$$(8.15)\quad w(x,\xi,\omega) = \sum_{\alpha=1}^{mk} \sum_{j=0}^{N} f_{j+m-1}(\phi^{\alpha}(x,\xi,\omega))a^{\alpha j}(x,\xi,\omega) + $$

$$+ h^{N}(x,\xi,\omega) ,$$

where $f_o(s) = \log_{-n}(s)$, h^N can be made as smooth as we like if N is taken large enough, and the functions $\phi^{\alpha}(x,\xi,\omega)$ satisfy the characteristic equation in x.[15]

Lemma 10.1, to be proved in Section 10, implies that $k(x,\xi)$ is a smooth function of x and ξ, except on the envelope with respect to ω of the surfaces $\phi^{\alpha}(x,\xi,\omega) = 0$, i.e. except if x lies on a ray which passes through ξ. The locus of points which can be joined to ξ by a ray is called the ray conoid with vertex ξ. The representation formula (8.8) immediately implies the

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[15] Note that the characteristic equations corresponding to a system and its adjoint are identical.

Generalized Huygens Principle:[16] The differentiability properties of the solution at a point ξ depend on the differentiability properties of the Cauchy data only at the intersection of the ray conoid through ξ with the initial hyperplane.

D. Caustics and the Riemann function.

The construction by means of progressing waves can be carried out only up to the first point where one of the phase surfaces $\phi^a(x,\xi,\omega) = 0$ develops a singularity, i.e. the first point of a caustic. If the principal part of the differential operator $\mathcal{L}$ has constant coefficients, then the surfaces $\phi^a(x,\xi,\omega) = 0$ are planes in x-space, and no caustic can occur. On the other hand, if the principal part of $\mathcal{L}$ has non-constant coefficients, then in general caustics arise. We could overcome this difficulty by the method of Section 11. However, the representation formula (8.8) also provides a method for extending solutions beyond caustics, and we shall use this method in order to extend the Riemann function itself beyond caustics.

Since the Riemann function is a solution of the differential equation, the Riemann function may be represented in terms of the Riemann function; it follows from

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[16] This principle was first enunciated by R. Courant and P. D. Lax [4].

(8.8) that

$$(8.16)\quad k(\xi_1,\xi_3) = \int P^o[k(\tau_2,\eta,\xi_3),\ k(\xi_1,\tau_2,\eta)]\, d\eta\ .$$

This fact is known as "Huygens' major premise,"[17] or the semi-group property. Provided that τ_1 and τ_2, τ_2 and τ_3 are sufficiently close to each other respectively, the integrand in (8.16) can be constructed by the method of progressing waves.

For the proof of the generalized Huygens' principle where caustics are involved, we cannot use Lemma 10.1 directly, but we can apply the same type of reasoning to (8.16). From (8.14) and (8.15), it follows that $k(\xi_i,\xi_{i+1})$ can be expressed in the form

$$(8.17)\quad k(\xi_i,\xi_{i+1}) = \int_{|\omega|=1}\left\{\sum_{\alpha=1}^{mk}\sum_{j=0}^{N} f_{j+m-1}(\phi^{\alpha}(\xi_i,\xi_{i+1},\omega))\times a^{\alpha j}(\xi_i,\xi_{i+1},\omega) + h^N(\xi_i,\xi_{i+1},\omega)\right\} d\omega.$$

Equations (8.16) and (8.17) show that the smoothness of $k(\xi_1,\xi_3)$ is determined by the smoothness of integrals of the form

$$(8.18)\quad J = \int\left\{\int_{|\omega_1|=1} f^{(1)}(\phi(\xi_2,\xi_3,\omega_1))a^{(1)}(\xi_2,\xi_3,\omega_1)\, d\omega_1 \times \int_{|\omega_2|=1} f^{(2)}(\phi(\xi_1,\xi_2,\omega_2))a^{(2)}(\xi_1,\xi_2,\omega_2)\, d\omega_2\right\} d\eta_2\ ,$$

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[17] See J. Hadamard [7], pp. 53-54, for a discussion of the various versions of Huygens' principle.

where $f^{(1)}(s)$ and $f^{(2)}(s)$ are C^∞ outside every neighborhood of the origin, and $a^{(1)}$ and $a^{(2)}$ are C^∞ functions. We denote the parameters in the integrand by $X = (\xi_1,\xi_3)$, and denote the variables of integration by $Y = (\xi_2,\omega_1,\omega_2)$. Then (8.18) may be abbreviated to

$$(8.19) \qquad J = \int f^{(1)}(\phi^{(1)}(X,Y))f^{(2)}(\phi^{(2)}(X,Y))a(X,Y)dY,$$

where the integration is taken over a certain manifold in Y-space. Let S^1 and S^2 be surfaces in Y-space given by the equations $\phi^{(1)} = 0$ and $\phi^{(2)} = 0$. If τ_1 and τ_2, τ_2 and τ_3 are sufficiently close to each other, S^1 and S^2 are regular surfaces. This is simply the requirement that there be no caustics for the Riemann functions in the integrand.

(a) If S^1 and S^2 are not tangent at any of their intersections, then we can introduce $\phi^{(1)}$ and $\phi^{(2)}$ as local coordinates in Y-space in the neighborhoods of S^1 and S^2 respectively. As in the proof of Lemma 8.1, we can introduce a C^∞ partition of unity in Y-space such that in the support of each function in the partition, either $\phi^{(1)} \neq 0$ or $\phi^{(1)}$ is a local coordinate, and either $\phi^{(2)} \neq 0$ or $\phi^{(2)}$ is a local coordinate. By integration by parts with respect to $\phi^{(1)}$ and $\phi^{(2)}$ in those neighborhoods where $\phi^{(1)}$ and $\phi^{(2)}$ are local coordinates, we can make the integrand as smooth as we like.

(b) If, on the other hand, the gradients $\phi_Y^{(1)}$ and $\phi_Y^{(2)}$ are parallel at some point in Y-space where S^1 and S^2 intersect, then since obviously $\phi^{(1)}_{\omega_2} \equiv \phi^{(2)}_{\omega_1} \equiv 0$, we must have $\phi^{(1)}_{\omega_1} = \phi^{(2)}_{\omega_2} = 0$. Hence ξ_2 lies on the ray conoid through ξ_3, and ξ_2 lies on the ray conoid through ξ_1. Moreover, since $\phi^{(1)}_{\xi_2}$ is parallel to $\phi^{(2)}_{\xi_2}$ at the point where S^1 and S^2 intersect, these conoids are tangent there. It follows from the theory of first-order partial differential equations that ξ_1 and ξ_3 lie on the same ray.

Hence the Riemann function is a smooth function of its arguments except if ξ_1 and ξ_3 lie on the same ray. This is the same as the generalized Huygens principle. A similar argument can be used to extend the generalized Huygens principle where an arbitrary number of intermediate points ξ_2, ξ_2', etc. are required.

We remark in passing that the same reasoning shows that singularities in the initial data can only propagate along characteristics. In fact, it would be possible to base the treatment of singularities upon the representation formula (8.8) instead of a direct calculation.

9. Preliminary Remarks about Caustics.

A. Definition and examples of caustics.

We have observed in Sections 3 and 5 that, in general, our formal construction cannot be carried out for all values of t. We shall consider the case where the construction is impossible because one or more of the phase surfaces ϕ^a = constant develop singularities, while the system (5.1) remains regular and hyperbolic. A connected set of points where a characteristic surface has such a singularity is called a caustic.

In order to examine an example of a caustic, let us consider the wave equation with $n = 2$. The characteristic surfaces for this equation can be characterized as developable surfaces whose rays are inclined at an angle of 45° to the planes $t = \text{const}$. Thus the right circular cone with vertex at the point $(1,0,0)$, whose intersection with the plane $t = 0$ is the unit circle, is a characteristic surface. This surface has a singularity at its vertex. It can be verified (see Lemma 9.2 in this section), that the ordinary differential equations (4.18) have a singular coefficient at the vertex. Consequently if, for the wave equation, we prescribe Cauchy data which has a jump discontinuity across the unit circle, then the magnitude of the jump in the solution becomes infinite at the vertex of the cone, and the formal construction breaks down. We may think of this phenomenon as a focusing of the singularity of the initial data. Other characteristic surfaces for the wave equation can be exhibited for which the caustic is a curve, namely those developables which have an edge of regression. In some cases, the caustic itself may have a cusp.

Caustics are always involved if we consider problems where the initial data has a "point singularity" (for example the n-dimensional δ-function), since the vertex of the ray conoid is a caustic point. If the coefficients of the principal part of the differential operator are variable, then in general the ray conoid will also have singularities away from the vertex. This phenomenon is sometimes called "focusing due to the medium." We shall see that there is no essential difference between caustics due to variable coefficients and caustics due to curvature of the curves bearing the singularity in the initial data.

B. Continuation of characteristic surfaces beyond caustics.

In spite of the fact that the characteristic equation is nonlinear, and that caustics may have irregular shapes, the rays which generate the characteristic surface can always be continued beyond the caustic. Hence the characteristic surface itself can be continued beyond the caustic. This fact is embodied in

Lemma 9.1: In the neighborhood of a particular ray which passes through a caustic, a characteristic surface S can be represented as the envelope of regular characteristic surfaces.

Proof: Let τ be the parameter along the ray defined in (4.16), and let $\tau = \tau_0$ at the caustic point. Let τ_1 be a number less than τ_0, and sufficiently close to τ_0. The intersection of S with the hyperplane $t = t(\tau_1)$ is regular. We call this intersection "C." C can be represented as the envelope of its tangent planes.[18] Let $S(\omega, \tau_1)$ be the family of characteristic surfaces whose intersections with the plane $t = \tau_1$ are the tangent planes to C. The theory of first order partial differential equations shows immediately that S is the envelope of $S(\omega, \tau_1)$ with respect to ω. Since the differential equation is regular at $x(\tau_0)$, the surfaces $S(\omega, \tau_1)$ are regular if τ_1 is sufficiently close to τ_0.

C. The behavior of generalized progressing wave solutions near a caustic.

We can obtain information about the behavior of the coefficients a^j in the neighborhood of a caustic by determining the nature of the singularity of the coefficients of the ordinary differential equations (4.21) satisfied by the scalars σ_β^j.

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[18] This representation is given analytically in the proof of Lemma 10.4 in the next section.

Lemma 9.2: If a caustic of a characteristic surface S intersects a ray which lies in S at $x(\tau_o)$, then along this ray, the coefficients $\rho^{\gamma\beta}$ are regular at the caustic, and

$$(7.1) \qquad P^{\gamma\beta}(\tau) = \rho^{\gamma\beta} \frac{\mu}{2(\tau-\tau_o)} + F^{\gamma\beta}(\tau) ,$$

where $F^{\gamma\beta}(\tau)$ is regular at $\tau = \tau_o$, and μ is a positive integer which depends upon the shape of S in the neighborhood of the caustic.

Since this lemma is not used in the sequel, we shall omit the proof. The proof appeared in D. Ludwig, Doctoral Dissertation, New York University, December 1958.

As an immediate consequence of this lemma, we obtain the following

Theorem: Along a ray which lies in S and passes through a caustic, the scalars $\sigma_\beta{}^o$ can be represented in the form

$\sigma_\beta{}^o = (\tau-\tau_o)^{-\mu/2} s_\beta(\tau)$, where $s_\beta(\tau)$ is regular at $\tau = \tau_o$.

Remark: The behavior of the coefficients $\sigma_\beta{}^{aj}$ $(j > 0)$ can be determined in a similar way from the inhomogeneous equations (4.21).

D. Summary.

We can draw the following conclusions from the discussion of this section:

1. The formal solution whose construction was

described in Sections 4 and 5 is, in general, not valid for all values of t, since caustics will generally arise and the coefficients $a^j(x)$ become infinite at caustics.

2. Nevertheless, the rays which generate the characteristic surfaces can be continued through the caustic, and the characteristic surfaces themselves can be continued through the caustic by taking envelopes of regular characteristic surfaces.

3. The behavior of the coefficients a^j near the caustic can be determined from the shape of the characteristic surfaces at the caustic.

10. Lemmas on Integrals and Envelopes.

In this section we shall derive some properties of integrals of distributions which depend upon parameters. The results of this section will make it possible for us to extend the formal solutions of Sections 4 and 5 beyond caustics, and to investigate the behavior of solutions at caustics.

We can summarize the results of this section as follows: Let $u(x,\xi)$ be a generalized progressing wave with the phase surfaces $S(\xi,c)$. Then $J(x) \equiv \int u(x,\xi)\, d\xi$ is a sum of generalized progressing waves whose phase surfaces are obtained by forming the envelope of $S(\xi,c)$ with respect to ξ. The wave form of $J(x)$ is obtained from the wave form of $u(x,\xi)$ by means of fractional integration. Conversely, certain generalized progressing waves $J(x)$ can be represented as a superposition of waves $u(x,\xi)$, where the corresponding phase surfaces $S(\xi,c)$ are planes.

A. Differentiability properties of integrals of waves.

As an example of a distribution depending upon parameters, let us consider the set of functions $H(\phi(x,\xi))$, where H is the Heaviside step function:

$$H(s) = 1 \quad \text{if} \quad s \geq 0 ,$$

$$H(s) = 0 \quad \text{if} \quad s < 0 .$$

Here $\phi(x,\xi)$ is a C^∞ function of the vectors x and ξ, $\phi_x(x) \neq 0$ if $\phi(x) = 0$. The equation $\phi(x,\xi) = 0$ defines a family of regular surfaces $S(\xi)$ in x-space, and the function $H(\phi(x,\xi))$ has a jump discontinuity across the surface $S(\xi)$. More generally we may consider the distribution $f(\phi)$, where $f(s)$ is a C^∞ function in the exterior of every neighborhood of the origin. Now we can formulate

Lemma 10.1: Let $a(x,\xi)$ be a C^∞ function of x and ξ with compact support in ξ-space for each value of x, let $\phi(x,\xi)$ have the properties described above, and let $J(x)$ be defined by

$$J(x) \equiv \int f(\phi(x,\xi))\, a(x,\xi)\, d\xi ,$$

where the integration is taken over all of ξ-space. Then $J(x)$ is a smooth function of x, except along the envelope of the surfaces $S(\xi)$.

Proof: Let $\sum(x)$ be the set of points in ξ-space such that $\phi(x,\xi) = 0$. If, for a particular value of x, $\phi_\xi \neq 0$ for all ξ in $\sum(x)$, then ϕ can be introduced as a coordinate in ξ-space in a neighborhood of $\sum(x)$. We can introduce a C^∞ partition of unity in ξ-space such that, in the support of every function in the partition, either ϕ is a local coordinate or $\phi \neq 0$. The smoothness properties of J depend only on those terms in the integral where ϕ is a local coordinate. Each of these terms may be written in the form

$$J_i = \iint f(\phi)\, a_i(x,\phi,\xi')\, d\phi\, d\xi' ,$$

where a_i has compact support. By repeated integration by parts with respect to ϕ, we can make the integrand have as many derivatives as we like. Hence J is infinitely differentiable.

If, on the other hand, $\phi_\xi = 0$ at some point on $\sum(x)$, then x lies on the envelope of the surfaces $S(\xi)$. Q.E.D.

B. Fractional integration and differentiation and their adjoints.

We can obtain more detailed information about the behavior of J in the neighborhood of the envelope, but it is first necessary to make some definitions and remarks: If $f(s)$ is a continuous function in the interval $-1 \leq s \leq 1$, then we can define integration of f to a fractional order by the equation:

$$(10.1) \qquad I^+(\alpha)\, f \equiv f_\alpha(s) = \frac{1}{\Gamma(\alpha)} \int_{-1}^{s} f(t)\,(s-t)^{\alpha-1}\, dt \qquad (\alpha > 0).$$

It can be verified that $I^+(\alpha)\, I^+(\beta)\, f = I^+(\alpha+\beta)\, f$. We define the transformation T by the equation:

$$(10.2) \qquad T\, f(s) = f(-s)\ .$$

We define another transformation of f by the equation:

$$(10.3) \qquad I^{-}(\alpha)\, f = T\, I^{+}(\alpha)\, T\, f\ ,$$

or

$$(10.4) \qquad I^{-}(\alpha)\, f = \frac{1}{\Gamma(\alpha)} \int_{s}^{1} f(t)\, (t-s)^{\alpha-1}\, dt\ .$$

Again, $I^{-}(\alpha)\, I^{-}(\beta)\, f = I^{-}(\alpha+\beta)\, f$. It can be verified immediately that if f and g are continuous then

$$(10.5) \qquad \int_{-1}^{1} f_{\alpha}^{+}(s)\, g(s)\, ds = \int_{-1}^{1} f(s)\, g_{\alpha}^{-}(s)\, ds\ .$$

Thus $I^{+}(\alpha)$ and $I^{-}(\alpha)$ are adjoint operations. Fractional differentiation and its adjoint can be defined by means of a fractional integration followed by a differentiation. Fractional differentiation and integration can be defined for distributions with support inside $(-1,1)$ by means of (10.5). For distributions which are arbitrarily smooth outside $(-1,1)$, fractional differentiation and integration may be defined modulo a smooth function by means of (10.5). $I^{+}(\alpha)$ and $I^{-}(\alpha)$ do not commute, but it can be verified that $[I^{+}(\frac{1}{2})\, I^{-}(\frac{1}{2}) - I^{-}(\frac{1}{2})\, I^{+}(\frac{1}{2})]\, f$ is a C^{∞} function if f is C^{∞} outside a closed subinterval of $-1 < s < 1$.

C. The wave form and phase function of the integral of a wave.

The following discussion is a nearly trivial extension of the method of stationary phase.

Let $f_o(s)$ be a distribution which is C^∞ outside a closed subinterval of $(-1,1)$. Let the phase function $\phi(x,\xi)$ be a C^∞ function of the vector x and the single parameter ξ. We assume that $\phi_x \neq 0$. Hence $\phi(x,\xi) = c$ is the equation of a family of regular surfaces $S(c,\xi)$ in x-space. We assume that ξ is determined as a single-valued function of x by the condition of stationary phase

$$\phi_\xi(x,\xi) = 0, \tag{10.6}$$

and write

$$\xi = \zeta(x).\tag{10.7}$$ [19]

We assume further that

$$\phi(x,\xi) = \phi(x,\zeta(x)) + (\xi-\zeta(x))^\beta \psi(x,\xi), \tag{10.8}$$

where $\psi(x,\zeta(x)) \neq 0$. The integer β may depend upon x. Let $\tilde{\phi}(x)$ be defined by

$$\tilde{\phi}(x) \equiv \phi(x,\zeta(x)). \tag{10.9}$$

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[19] If we allow $\zeta(x)$ to be multivalued, slight modifications in the following discussion are required.

The surface $S(c)$, given by setting $\tilde{\phi}(x) = c$, is the envelope of the surfaces $S(c,\xi)$. $S(c)$ is regular if $\phi_{\xi\xi}(x,\zeta(x)) \neq 0$, i.e. if $\beta = 2$. Let $a(x,\xi)$ be a C^{∞} function with compact support.

Lemma 10.2: Let $J(x)$ be defined by

$$(10.10) \qquad J(x) \equiv \int f_o(\phi(x,\xi))\, a(x,\xi)\, d\xi \, .$$

We consider three cases. Let $\pm$ denote the sign of $\psi(x,\zeta)$, and $\mp$ denote its negative.

(a) If β is even, then

$$(10.11) \qquad J = \sum_{j=0}^{N} f^{\mp}_{\frac{j+1}{\beta}} (\tilde{\phi}(x))\, \tilde{a}^{j}(x) + h^{N}(x) \, ,$$

and

$$(10.12) \qquad \tilde{a}^{o}(x) = \pm \frac{2\Gamma(1/\beta)}{\beta} |\psi(x,\zeta|^{-\frac{1}{\beta}} a(x,\zeta) \, .$$

(b) If β is odd, then

$$(10.13) \qquad J = \sum_{j=0}^{N} [f^{+}_{\frac{j+1}{\beta}}(\tilde{\phi}(x))\, \tilde{a}^{+j}(x) + f^{-}_{\frac{j+1}{\beta}}(\tilde{\phi}(x))\, \tilde{a}^{-j}(x)]$$

$$+ h^{N}(x) \, ,$$

and

$$(10.14) \qquad \tilde{a}^{+o}(x) = \tilde{a}^{-o}(x) = \pm \frac{\Gamma(1/\beta)}{\beta} |\psi(x,\zeta|^{-\frac{1}{\beta}} a(x,\zeta) \, .$$

(c) If $\phi(x,\xi) \equiv \tilde{\phi}(x)$, $(\beta = \infty)$, then it follows at once that

$$(10.15) \qquad J = f_o(\tilde{\phi}(x))\, \tilde{a}(x) \, ,$$

and

$$(10.16) \qquad \tilde{a}(x) = \int a(x,\xi)\, d\xi \ .$$

In any case, $\tilde{a}^j(x)$ are C^∞, and the remainder $h^N(x)$ can be made as smooth as we like if N is taken large enough.

Proof: It follows from (10.8) that

$$(10.17) \qquad \phi_\xi(x,\xi) = \beta(\xi-\zeta)^{\beta-1} \,[\psi(x,\xi)+\xi-\zeta) \text{ reg. funct.}],$$

and

$$(10.18) \qquad \frac{1}{\phi_\xi(x,\xi)} = \frac{(\xi-\zeta)^{1-\beta}}{\beta\ \psi(x,\zeta)}\,[1+ (\xi-\zeta) \text{ reg. funct.}] \ .$$

Equation (10.8) also implies that

$$(10.19) \qquad (\xi-\zeta) = \left|\frac{\phi-\tilde{\phi}}{\psi(x,\xi)}\right|^{1/\beta} \operatorname{sgn}\,(\xi-\zeta) \ ,$$

and hence

$$(10.20) \qquad \frac{1}{\phi_\xi(x,\xi)} = \frac{(\operatorname{sgn}(\xi-\zeta))}{\beta\ \psi(x,\zeta)}^{1-\beta} \left|\frac{\phi-\tilde{\phi}}{\psi(x,\zeta)}\right|^{\frac{1-\beta}{\beta}} [1+(\xi-\zeta)\text{reg. funct.}]$$

Now, introducing ϕ as a new variable of integration, we have

$$(10.21) \qquad J_1 = \int_{\phi(x,-\infty)}^{\phi} (\operatorname{sgn}(\xi-\zeta)^{1-\beta} f(\phi)\, \frac{a_1(x,\xi)}{\beta\ \psi} \left|\frac{\phi-\tilde{\phi}}{\psi}\right|^{\frac{1-\beta}{\beta}} d\phi \ ,$$

$$(10.22) \qquad J_2 = \int_{\phi}^{\phi(x,\infty)} (\operatorname{sgn}(\xi-\zeta))^{1-\beta} f(\phi)\, \frac{a_2(x,\xi)}{\beta\ \psi} \left|\frac{\phi-\tilde{\phi}}{\psi}\right|^{\frac{1-\beta}{\beta}} d\phi \ ,$$

where $a_1(x,\xi)$ and $a_2(x,\xi)$ are regular functions. We note that

$$a_1(x,\zeta) = a_2(x,\zeta) = a(x,\zeta) .$$

Now we assume that β is even and $\psi(x,\zeta) > 0$; the proof for the other cases is obtained by obvious modification. Since $\psi(x,\zeta) > 0$ and β is even, $\phi(x,\infty)$ and $\phi(x,-\infty)$ must both be greater than $\tilde{\phi}$. Nothing is changed if we set $a_1(x,\xi) \equiv 0$ for $\phi \geq \phi(x,-\infty)$ and $a_2(x,\xi) \equiv 0$ for $\phi \geq \phi(x,\infty)$. Thus we can extend the limits of integration to the interval $\tilde{\phi} \leq \phi \leq 1$. Combining J_1 and J_2 ,

$$(10.23) \quad J = 2 \int_{\tilde{\phi}}^{1} \frac{f(\phi)(\phi-\tilde{\phi})^{(1/\beta)-1}}{\beta(\psi(x,\zeta))^{1/\beta}} (a_1(x,\xi) + a_2(x,\xi) \, d\phi .$$

Now we expand the integrand in a power series in ϕ:

$$(10.24) \quad \frac{2}{\beta\psi^{1/\beta}} (a_1(x,\xi) + a_2(x,\xi) = \sum_{j=0}^{N} \frac{\tilde{a}^j(x)(\phi-\tilde{\phi})^{j/\beta}}{\Gamma(\frac{j+1}{\beta})} + H^N(x,\phi)(\phi-\tilde{\phi})^{(N+1)/\beta} .$$

Substituting (10.24) into (10.23), we have

$$(10.25) \qquad J(x) = \sum_{j=0}^{N} f^{-}_{\frac{j+1}{\beta}}(\tilde{\phi}) \, \tilde{a}^j(x) + h^N(x) ,$$

which was to be proved.

Lemma 10.2 can be used to prove the corresponding results when ξ is a vector : $\xi = (\xi', \ldots, \xi^k)$. The most important case is when $\tilde{\phi}(x)$ is a regular envelope of $\phi(x,\xi)$, i.e. when $\beta = 2$. For this case, the more general result may be stated as follows:

Lemma 10.3: Let $\xi = (\xi^1,\ldots,\xi^k)$, and let $f(s)$, $\phi(x,\xi)$, $\zeta(x)$ and $a(x,\xi)$ satisfy the hypotheses of Lemma 8.2. If, in addition,

$$(10.26) \qquad \phi_{\xi^i\xi^i}(x,\zeta(x)) < 0 \qquad i = 1,\ldots,\ell,$$

$$(10.27) \qquad \phi_{\xi^i\xi^i}(x,\zeta(x)) > 0 \qquad i = \ell+1,\ldots,k ,$$

then $J(x)$, given by the integral

$$(10.28) \qquad J(x) = \int \sum_{j=o}^{\infty} f_j(\phi(x,\xi))\, a^j(x,\xi)\, d\xi ,$$

has the form of a sum of waves plus a remainder. The leading term in the expansion of J is

$$(10.29) \qquad F(\phi(x,\zeta(x))\, \tilde{a}(x) ,$$

where

$$(10.30) \qquad F = I^-\left(\frac{k-\ell}{2}\right) I^+\left(\frac{\ell}{2}\right) f ,$$

and

$$(10.31) \qquad \tilde{a}(x) = \frac{(-1)^{\ell}(2\pi)^{k/2}\, a^o(x,\zeta(x))}{\sqrt{(-1)^{\ell}\prod_{i=1}^{k} \phi_{\xi^i\xi^i}(x,\zeta)}} .$$

Proof: The lemma follows immediately from k times application of Lemma 10.2.

D. Representation of a family of surfaces as the envelope of a family of planes.

With the aid of Lemma 10.3 it is easy to prove a converse to Lemma 10.3, namely a representation of certain waves as a superposition of plane waves. First we must show how a family of surfaces whose principal curvatures are nowhere zero can be represented as the envelope of a family of planes.

Lemma 10.4: Let $x = (x^1, \ldots, x^n) = (x^1, \ldots, x^{n-1}, s) = (y, s)$, and let $\xi = (\xi^1, \ldots, \xi^{n-1})$. Let $\phi(x)$ be a C^∞ function of x such that grad $\phi(x) \neq 0$. Then $\phi(x) = c$ is the equation of a family of surfaces in x-space. If none of the surfaces $S(c)$ has a zero principal curvature, then there exists a function $\phi(x, \xi)$ such that

(1) For fixed ξ and c, the surface $S(c, \xi)$ given by $\phi(x, \xi) = c$ is a plane.

(2) The planes $S(c, \xi)$ envelope $S(c)$.

(3) $\phi_{\xi^i \xi^i} \neq 0$ at the envelope $(i = 1, \ldots, n-1)$.

Proof: Since grad $\phi \neq 0$, we can assume without loss of generality that $\phi_{x^n}(x^1, \ldots, x^n) \neq 0$. Hence by solving the equation

$$(10.32) \qquad \phi(x^1, \ldots, x^n) = c,$$

we can determine x^n as a function of $(x^1, \ldots, x^{n-1}, c)$:

$$(10.33) \qquad x^n = g(x^1, \ldots, x^{n-1}, c).$$

Equation (10.33) can also be written

$$(10.34) \qquad s = g(y,c) .$$

Since we have assumed that none of the principal curvatures of $S(c)$ are zero, the matrix

$$(g_{ij}) = \left(\frac{\partial^2 g}{\partial x^i \partial x^j}\right) \qquad i,j = 1,\ldots,n-1 ,$$

must be non-singular. By a rotation of coordinate axes, we can insure that $g_{ii} \neq 0$ $(i = 1,\ldots,n-1)$.

We define $G(x,\xi,c)$ by the equation

$$(10.35) \qquad G(x,\xi,c) \equiv x^n - g(\xi,c) - \sum_{i=1}^{n-1} g_i(\xi,c)(x^i-\xi^i) ,$$

and define $S(c,\xi)$ by setting

$$(10.36) \qquad G(x,\xi,c) = 0.$$

The surfaces $S(c,\xi)$ are planes in x-space which are tangent to $S(c)$ at the point $(\xi^1,\ldots,\xi^{n-1},g(\xi,c))$. The equations for the envelope are

$$(10.37) \qquad \frac{\partial G}{\partial \xi^j} \equiv -\sum_{i=1}^{n-1} g_{ij}(x^i-\xi^i) = 0 \qquad j = 1,\ldots,n-1.$$

Hence $\xi^i = x^i$ $(i = 1,\ldots,n-1)$ at the envelope, and the equation of the envelope is

$$(10.38) \qquad x^n - g(x^1,\ldots,x^{n-1},c) = 0 .$$

In order to complete the proof of the lemma, it is only necessary to write the equation for $S(c,\xi)$ in implicit form, and verify the condition for regularity of the envelope. Differentiation of (10.35) shows that

$$(10.39) \qquad \frac{\partial G}{\partial c}(x,\xi,c) = -\frac{\partial g}{\partial c} - \sum_{i=1}^{n-1} \frac{\partial g_i}{\partial c}(x^i-\xi^i).$$

Since $\partial g/\partial c = 1/\phi_{x^n}$, $\partial g/\partial c \neq 0$; consequently $\partial G/\partial c \neq 0$ if $\sum_{i=1}^{n-1}(x^i-\xi^i)^2$ is sufficiently small. Thus we can determine c as a function of (x,ξ) from (10.36):

$$(10.40) \qquad \phi(x,\xi) = c .$$

A short calculation shows that, at the envelope,

$$(10.41) \qquad \phi_{\xi^j\xi^j} = \frac{g_{jj}}{g_c} \qquad j = 1,\ldots,n-1 .$$

By hypothesis, $g_{jj} \neq 0$, and hence $\phi_{\xi^j\xi^j} \neq 0$ $(j = 1,\ldots,n-1)$. Q.E.D.

E. Representation of certain waves as a superposition of plane waves.

Now we are in a position to show how a distribution in the form of a wave

$$(10.42) \qquad u(x) = \sum_{j=0}^{N} F_j(\tilde{\phi}(x))\,\tilde{a}^j(x) ,$$

can be represented as a superposition of plane waves.

Lemma 10.5: Let $F_o(s)$, $\tilde{\phi}(x)$ and $\tilde{a}^j(x)$ satisfy the following conditions:

(1) $F_o(s)$ is C^∞ outside a subinterval of $(-1,1)$

(2) $\phi(x)$ is a C^∞ function, $\phi_{x^n} \neq 0$ for all x

such that $|\tilde{\phi}(x)| \leq 1$.

(3) $\tilde{a}^j(x)$ are C^∞ functions with compact support.

(4) The principal curvatures of the surfaces $\tilde{\phi}(x) = c$ are different from zero for all x such that $|\tilde{\phi}(x)| \leq 1$. ℓ of these principal curvatures have their sign different from the sign of ϕ_{x^n}.

Then $u(x)$ may be represented as a superposition of plane waves, plus remainders. Specifically, u may be represented as an integral over $(n-1)$-dimensional ξ-space of waves with the plane phase surfaces $\phi(x,\xi) = \text{const}$. The wave form of the leading term in the integrand is given by

$$(10.43) \qquad f = I^-\left(-\frac{n-1-\ell}{2}\right) I^+\left(-\frac{\ell}{2}\right) F \ .$$

Proof: Let $\phi(x,\xi)$ be defined as in the proof of Lemma 10.4. Let $a^0(x,\xi)$ be a C^∞ function with compact support inside the region where $\phi(x,\xi)$ is defined, such that

$$(10.44) \qquad a^0(x^1,\ldots,x^n;x^1,\ldots,x^{n-1}) =$$

$$= \frac{(-1)^\ell \sqrt{(-1)^\ell \prod_{i=1}^{n-1} \phi_{\xi^i\xi^i}}}{(2\pi)^{\frac{n-1}{2}}} \ \tilde{a}^0(x) \ .$$

Lemma 10.3 implies that

$$(10.45) \quad \int f_0(\phi(x,\xi))a^0(x,\xi)d\xi = F_0(\tilde{\phi}(x))\tilde{a}^0(x) + \ldots$$

where ... represents smoother terms. The next term in the expansion can be handled in a similar way, and so on.

11. Extension of Solutions Beyond Caustics.

We observed in Section 9 that the generalized progressing wave solutions described in Sections 4, 5 and 6 do not in general exist beyond the first point where a caustic is present. However, as we saw in Section 9, the family of phase surfaces of the wave can be extended beyond caustics by taking envelopes of regular characteristic surfaces. This fact is closely connected with the theory of the complete integral for first order partial differential equations (see R. Courant and D. Hilbert [1], Chapter II). The lemmas of Section 10 show that generalized progressing wave solutions have an analogous property; integrating a generalized progressing wave with respect to certain parameters ξ corresponds to taking the envelope of the phase surfaces $\phi(x,\xi) = \text{const.}$ with respect to ξ. Thus, with the aid of Lemma 10.5, we can represent the solution as a superposition of regular waves even where caustics are present. In the region beyond the caustic, the solution can be calculated as in Section 4 by solving certain ordinary differential equations along the rays.

In order to illustrate the method for handling caustics, let us first consider a simple example. We assume that initial data for the 2-dimensional wave equation is given which is symmetric about the origin. Then the solution will be a function of t and r $(=\sqrt{x^2+y^2})$

alone. We assume further that this data is singular in the region $1 + \varepsilon \leq r \leq 2-\varepsilon$. This singularity will propagate along the cones $r \pm t = \text{const.}$, according to the theory of Sections 2 - 5. There is no difficulty with the wave with phase surfaces $r-t = \text{const.}$, since these surfaces are regular for $t > 0$. On the other hand, the surface $r+t = 1$ is singular for $t = 1$, and there the theory breaks down. In fact, the set of points given by $r = 0$, $1 \leq t \leq 2$ is a caustic of the family $r+t = c$ $(1 \leq c \leq 2)$. (See Figure 1.)

Now, according to Lemma 10.5, the initial data can be represented as a superposition of plane waves. The planes in question are tangent to the cones $r \pm t = \text{const.}$ There is a corresponding decomposition of the solution into plane waves. Using the method of stationary phase (Lemma 10.2), we can express the solution as a sum of progressing waves. Lemma 10.2, also shows that, on the caustic, the wave form of the leading term of the solution is a 1/2-derivative of the wave form of the data. Which 1/2-derivative it is depends upon the sign of $\phi_{\xi\xi}(x,\xi)$ on the cones $r+t = \text{const.}$ before the caustic. After passing through the caustic, the wave form of the leading term is altered by the transformation $I^{+}(\frac{1}{2})\, I^{-}(-\frac{1}{2})$, or $I^{-}(\frac{1}{2})\, I^{+}(-\frac{1}{2})$. Roughly speaking, this a 1/2 differentiation followed by a 1/2 integration, and the differentiability of the solution is essentially unchanged.

The following discussion is merely an extension of the method outlined above to a more general situation.

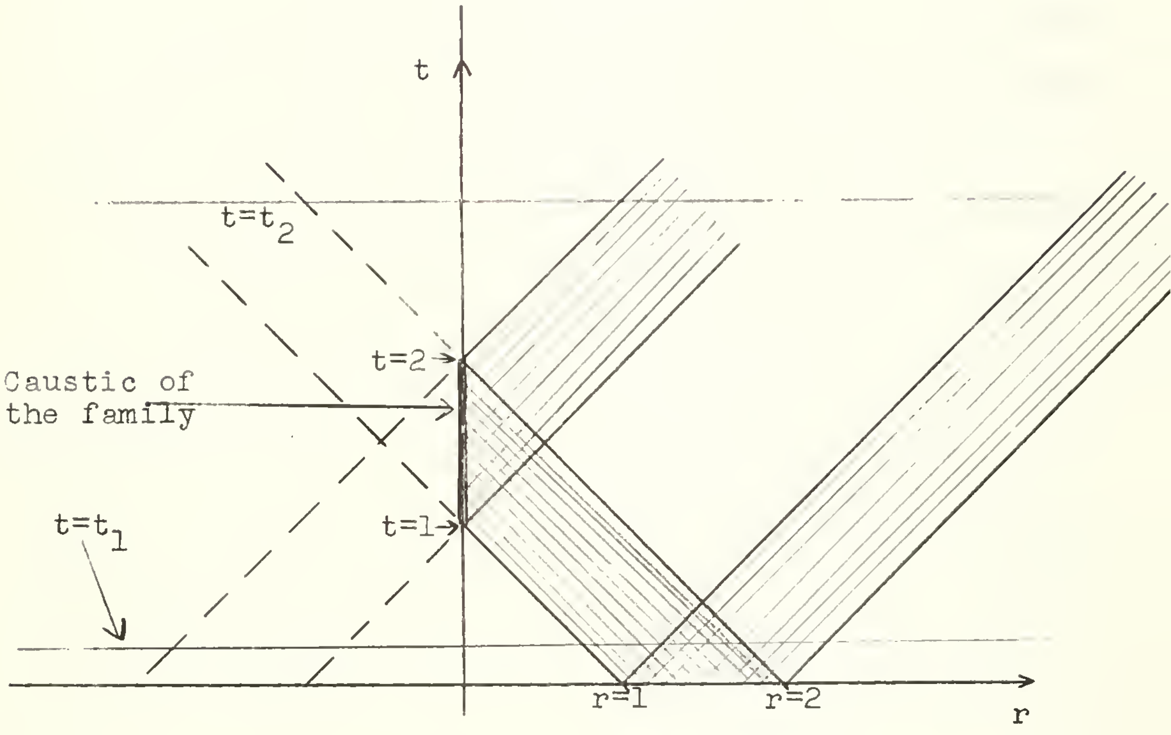

Figure 1

It is sufficient for our purpose to be able to carry out the construction in the neighborhood of a ray passing through a point of the caustic. Consider a family of characteristic surfaces $S(c)$ given by setting $\phi(x) = c$, $(c^2 \leq 1)$. We assume that the whole family is regular for $t = t_1$ and for $t = t_2$, but the caustics are present in the region given by $t_1 < t < t_2$.

We also assume that t_2 is near t_1, i.e. that we can represent the surfaces $S(c)$ as the envelopes of a family of characteristic surfaces which are regular in the region $t_1 \leq t \leq t_2$. An obvious way to do this would be to represent the intersection of $S(c)$ and the surface $t = t_1$ as an envelope of planes, and then continue this family of planes for $t > t_1$ as regular characteristic surfaces. We proved Lemma 10.4 with this purpose in mind. Lemma 10.4 states that any regular surface can be represented as an envelope of planes, provided that none of its principal curvatures are zero. However, the intersection of $S(c)$ and $t = t_1$ may have a zero principal curvature. For example, the converging cylinders $\sqrt{x^2+y^2} + t = c$ in (t,x,y,z)-space have this property. But caustics occur only when certain principal curvatures of $S(c)$ become infinte. These principal curvatures will certainly be different from zero near the caustic. Thus, perhaps using fewer than n-1 parameters, we may represent $S(c)$ as the envelope of regular characteristic surfaces. In analytic language, Lemma 10.4 implies that we can introduce parameters $\xi = (\xi^1,\ldots,\xi^p)$ and a function $\psi(y,\xi)$ such that the intersection of $S(c)$ with the hyperplane $t = t_1$ is the envelope of the surfaces given by $\psi(y,\xi) = c$.

Now we can extend the function $\psi(y,\xi)$ to functions $\psi^a(x,\xi) = \psi^a(t,y,\xi)$ $(a = 1,\ldots,mk)$, satisfying the characteristic equation (5.3a). This amounts to solving the ordinary differential equations (5.5) along the rays. The theory of first-order partial differential equations shows that $S(c)$ is the envelope of certain of the surfaces $S^{a}(c,\xi)$ given by setting $\psi^{a}(x,\xi) = c$. Moreover, t_1 and t_2 have been chosen in such a way that $S^{a}(c,\xi)$ are regular surfaces in the region $t_1 \leq t \leq t_2$.

Assume that we have a solution of (4.1) given by

$$(11.1) \qquad u(x) = \sum_{j=0}^{N} F_j(\emptyset(x))\, a^j(x) + h^N(x) ,$$

where $u(x)$ is uniformly smooth outside the region $|\emptyset(x)| < 1$. In the form (11.1), this solution is only valid in the region below the first caustic of the family $S(c)$. Lemma 10.5 implies that we can represent $u(t_1,y)$ as a superposition of plane waves. The leading term in the integrand has the wave form

$$(11.2) \qquad f = I^-\left(\frac{\ell-p}{2}\right) I^+\left(-\frac{\ell}{2}\right) F_0 .$$

Here ℓ is the signature of the matrix $\psi_{\xi^i\xi^i}(y,\zeta(y))$ $(i,j \leq p)$. Each of the waves in the integrand can be extended as a solution of the differential equation to the region $t_1 \leq t \leq t_2$.

The uniqueness theory shows that u is singular only along $S(c)$. We can express $u(t_2,y)$ as a sum of progressing waves by applying the method of stationary phase

(Lemma 10.3).

It follows that the leading term in the expansion of u has the wave form

$$(11.3) \qquad \tilde{F} = I^{-}\left(\frac{p-\ell'}{2}\right) I^{+}\left(\frac{\ell'}{2}\right) I^{-}\left(\frac{\ell-p}{2}\right) I^{+}\left(-\frac{\ell}{2}\right) F_o ,$$

where ℓ' is the signature of the matrix $\psi_{\xi^i\xi^i}(t_2, y, \zeta(y))$ $(i, j \leq p)$. Since in general $\ell \neq \ell'$, the wave form at $t = t_2$ is different from the wave form at $t = t_1$. However, since $\tilde{F}$ is obtained from F_o by $p/2$ differentiations ollowed by $p/2$ integrations, $\tilde{F}$ has essentially the same differentiability properties as F_o.

Lemma 10.2 can be applied p times to obtain an expansion of u which is valid at the caustic. At the caustic, the wave form will be a certain fractional derivative of F_o. At a perfect focus (as in Figure 1) the wave form is given by f, which is obtained by p 1/2 differentiations, and we may say that u loses p/2 derivatives at the caustic. At other types of caustics, less than p/2 derivatives will be lost.

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